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A discussion of bisexual populations with Wolbachia infection as an evolution algebra

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ABSTRACT

In this paper, Wolbachia infection in a bisexual and diploid population with a fixed cytoplasmic incompatibility rate w and maternal transmission rate d is studied as an evolution algebra. As the cytoplasmic incompatibility (CI) of the population causes deaths in the offspring, the evolution algebra of this model is not baric, and is a dibaric algebra if and only if the cytoplasmic incompatibility rate w is 1 and $d = 1$. The idempotent elements are given in terms of d and w . Moreover, this algebra has no absolute nilpotent elements when CI expression $w \neq 1$.

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1. Introduction

The manipulation of the host production by microbes is studied extensively in evolutionary biology and one particular type is a parasite called Wolbachia [4]. It is primarily found in insects and can be transferred to offspring, causing the mortality of the embryo of an infected male and an uninfected female. There are studies in the literature on Wolbachia-infected insects such as the terrestrial isopods [1], the honeybees [9]; and the mosquitoes [10]. In Singapore, the release of male Wolbachia-infected mosquitoes reduced the dengue mosquitoes by causing mortality of the uninfected dengue mosquitoes and consequently reduced the dengue disease incidences. Wolbachia-infection has two different effects, vector-competency of dengue transmission is diminished by the strength of Wolbachia-infection; the second one is the appearance of cytoplasmic incompatibility in the population. *Cytoplasmic incompatibility (CI)* is the reproductive incompatibility between males infected with a particular strain of bacteria and females not infected with this strain.

There are two different mathematical approaches to the discussion of the Wolbachia populations. The first approach considers the infected population as a time-discrete dynamical system [8]. This article addresses the second approach namely, considering the Wolbachia infected population as an evolution algebra. The evolution algebra of bisexual populations is studied by Ladra and Rozikov in [7]. The Wolbachia-infected populations considered in this paper are bisexual and also diploid. Hence, Wolbachia-infection of bee populations is not considered in this paper as male bees are haploid. Moreover, there are different strains of Wolbachia bacteria apparent in biological systems, hence it might be interesting to study higher dimensional evolution algebras of Wolbachia-infected populations.

However, within the scope of this paper, EABP is considered to have the minimum dimension, i.e. it is four-dimensional.

The theory of evolution algebras dates back to Mendel and a concise approach to the topic is given in the expository article in [3]. The mathematical theory of bisexual evolution algebras (EABP) is already established in [7], hence Wolbachia-infected population will be a particular example to implement the results known, and characterize its algebraic properties, list the similarities and the differences with [7]. The Wolbachia-infected population considered in this article is a four-dimensional evolution algebra of a bisexual population denoted by \mathcal{W} . The basis consists of two types of females f_1, f_2 and two types of males m_1, m_2 . Type 1 is a non-infected individual and type 2 is a Wolbachia-infected individual (denoted by superscript +), namely $f_1 = XX, f_2 = XX^+, m_1 = XY, m_2 = XY^+$.

The evolution algebra of a bisexual population with a Wolbachia infection, \mathcal{W} , shares the same properties as an EABP (Theorem 4.3). However, the Wolbachia-infected population reproduction causes deaths in the offspring, hence there are some noteworthy differences. The bisexual evolution algebra (EABP) is not a baric algebra, it is a dibaric algebra [7, Theorems 5.1 and 6.3]. On the other hand, \mathcal{W} is not a baric algebra (Theorem 4.4), and it is a dibaric algebra if and only if $w = 1 = d$ (Theorem 4.5).

The outlay of the paper will be as follows: Section 2 starts with the preliminaries from biology and evolution algebras. Section 3 mimics the known results on a bisexual evolution algebra in [7] to the case of a Wolbachia-infected bisexual population. Bustamante, Mellon and Velasco in [2], propose a method to determine whether a genetic algebra is an evolution algebra. Hence, using this method, it is shown that the Wolbachia-infected bisexual population is not an evolution algebra in Section 4. Section 5 studies the idempotent and absolute nilpotent elements of \mathcal{W} . Theorem 5.1 lists the idempotent elements and agrees with the results in [8, Section 3]. Moreover, Theorem 5.2 shows that there are no absolute nilpotent elements when CI rate is not 1.

2. Preliminaries

2.1. Biology

Definition 2.1. [4] *Cytoplasmic incompatibility (CI)* is the reproductive incompatibility between males infected with a particular strain of bacteria and females not infected with this strain.

Wolbachia is a particular example of a parasite in insects that is transmitted via reproduction. Although there are different strains of Wolbachia-infections in nature, to minimize the dimension of evolution algebra studied in this paper, only one strain of Wolbachia-infection is considered.

The cytoplasmic incompatibility of the population is given as w , and d is the probability of the transmission of the Wolbachia infection from a female to its offspring. In biological models, w is also known as the paternal affection rate and d is the maternal transmission rate (e.g. [5]). If both $w = d = 1$, the infected individual (male/female) produces all infected alleles. Mathematically speaking, if $w = 0$, then the infected male produces no infected alleles and if $d = 0$, then the infected female does not produce any infected gametes. Both $w = 0$ or $d = 0$ scenarios are not biologically observed, and not interesting. Hence, throughout the text, assume $w, d \in (0, 1]$.

The infected female/male produces gametes that are both infected and uninfected. The zygote from an infected male gamete and an uninfected female gamete is viable, hence there are deaths in the offspring population.

Table 1. Gamete crossing in a Wolbachia-infected population.

Gamete crossing	X	X^+	Y	Y^+
X	uninfected female	(death) no offspring	uninfected male	(death) no offspring
X^+	infected female	infected female	infected male	infected male

2.2. Evolution algebra

In this subsection, the necessary terminology, and historical remarks are quoted from [3], which is a brief and concise summary of the history of evolution algebras.

Interpretation of sexual reproduction laws of inheritance with algebraic symbols dates back to Mendel. More precise studies by Serebrowski, Kostitzin, Glivenkov, and Etherington (to name a few pioneers of the subject) gave rise to the term genetic algebras. Algebraic properties of special genetic algebras are studied by several authors.

Etherington also defined baric algebra in 1939 as a special genetic algebra.

Definition 2.2. A *character* for an algebra A is a nonzero multiplicative linear form on A , that is, a nonzero algebra homomorphism from A to \mathbb{R} . Not every algebra admits a character. For example, an algebra with zero multiplication has no character. A pair (A, σ) consisting of an algebra A and a character σ on A is called a *baric algebra*.

In 1970, Holgate introduced the notions of sex differentiation algebra (Definition 2.3) and dibaric algebra (Definition 2.4).

Definition 2.3. [7, Definition 6.1] Let $\mathcal{U} = \langle W, M \rangle_{\mathbb{R}}$ denote a two-dimensional commutative algebra over \mathbb{R} with the multiplication table

$$W^2 = M^2 = 0, \quad WM = \frac{1}{2}(W + M).$$

Then \mathcal{U} is called the *sex differentiation algebra*.

Notice that $\mathcal{U}^2 = \text{span}\{zt \mid z, t \in \mathcal{U}\} = \langle W + M \rangle_{\mathbb{R}}$ is an ideal of \mathcal{U} .

Definition 2.4. If an \mathbb{R} -algebra A admits a homomorphism onto the sex differentiation algebra then A is called a *dibaric algebra*.

Holgate also proved that if A is a dibaric algebra, then A^2 is a baric algebra.

The evolution algebras were introduced by Tian in his Ph.D. thesis to model the self-reproduction rules of non-Mendelian genetics in 2004.

Definition 2.5. Let I be an index set and E be a vector space over a field \mathbb{K} , with a basis $B = \{e_i \mid i \in I\}$ such that $e_i e_j = 0$ whenever $i \neq j$ and $e_i e_i = \sum_{k \in I} \omega_{ki} e_k$. Then E is called an *evolution algebra* over \mathbb{K} and B is a natural basis of E . The scalars $\omega_{ki} \in \mathbb{K}$ are the *structure constants* of A relative to B , the matrix $M_B := (\omega_{ki})$ is the *structure matrix* of A relative to B . Every evolution algebra is uniquely determined by its structure matrix.

Assume that a population consists of m different genetic types and consider the m -tuple $x = (x_1, \dots, x_m)$. Each component x_i of x denotes the probability that a random individual in the population belongs to the species that is determined by i th genetic type, hence $x_i \geq 0$ and $\sum_{i=1}^m x_i = 1$. Let

$x^0 = (x_1^0, \dots, x_m^0)$ be the probability distribution of species in the initial generations, and $P_{ij,k}$ be the probability that individuals in the i th and j th species interbreed to produce an individual k . Then the probability distribution $x' = (x'_1, \dots, x'_m)$ of the species in the first generation can be found by the total probability i.e.

$$x'_k = \sum_{i,j=1}^m P_{ij,k} x_i^0 x_j^0, \quad k = 1, \dots, m$$

where the cubic matrix $P = (P_{ij,k})_{i,j,k=1}^m$ satisfies the following conditions

$$P_{ij,k} \geq 0, \quad \sum_{k=1}^m P_{ij,k} = 1, \quad i, j \in \{1, \dots, m\}.$$

Bernstein defined the term *quadratic stochastic operator* (QSO) as a map $V : S^{m-1} \rightarrow S^{m-1}$ where

$$S^{m-1} = \left\{ (x_1, \dots, x_m) \in \mathbb{R}^m \mid x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\}$$

with $x^0 \mapsto x'$. This map V is the evolutionary operator that describes the inheritance process of a free population with m different genetic types. The evolution algebra of a bisexual population (EABP) is described in [7]. When the population is bisexual, the basis is partitioned into a set of females with different types indexed by $\{1, 2, \dots, n\}$, and the set of male types indexed by $\{1, 2, \dots, \nu\}$. The dimension of the population is the sum of the male and female types, that is $n + \nu$. The population is described by its state vector $(x, y) \in S^{n-1} \times S^{\nu-1}$, the product of two unit simplexes in \mathbb{R}^n and \mathbb{R}^ν respectively. Vectors x and y are the probability distributions of the females and males over the possible types which satisfy the equations:

$$x_i \geq 0, y_j \geq 0, \text{ for all } i \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, \nu\}$$

$$\sum_{i=1}^n x_i = 1 \text{ and } \sum_{i=1}^{\nu} y_i = 1.$$

Let $P_{ik,j}^{(f)}$ and $P_{ik,l}^{(m)}$ be inheritance coefficients defined as the probability that female offspring is type j and, respectively, that a male offspring is of type l when the parental pair is ik ($i, j = 1, \dots, n$; and $k, l = 1, \dots, \nu$). We have

$$P_{ik,j}^{(f)} \geq 0, \quad \sum_{j=1}^n P_{ik,j}^{(f)} = 1; \quad P_{ik,l}^{(m)} \geq 0, \quad \sum_{l=1}^{\nu} P_{ik,l}^{(m)} = 1 \tag{1}$$

3. Evolution algebra model of a Wolbachia infected population

The study of the Wolbachia-infected population as an evolution algebra is a particular example of the bisexual population. Wolbachia-infected populations considered in this paper do not form a model for an evolution algebra in the sense of Definition 2.5 as shown in Theorem 4.1. However, the evolution algebra of a bisexual population [7] is a suitable model. Most of this work is an interpretation of this problem with respect to the paper of Ladra and Rozikov [7].

Consider a population with four types of individuals, males without Wolbachia infection: XY , males with Wolbachia infection: XY^+ , females without Wolbachia infection: XX and females with Wolbachia infection: XX^+ where the basis elements of the population are denoted by $f_1 = XX, f_2 = XX^+, m_1 = XY, m_2 = XY^+$.

Define \mathcal{W} as the vector space generated by four basis elements $\mathcal{B} = \{f_1, f_2, m_1, m_2\}$ with the maternal transmission rate is given as d and the cytoplasmic incompatibility of the population CI rate is given as w . Stated differently, the type 2 (Wolbachia-infected) female individual (f_2) produces infected gamete with probability d and type 2 (Wolbachia infected) male individual (m_2) produces infected gamete with probability w (w is also called the paternal affection rate in literature [5]). An individual infected female produces an infected gamete X^+ with probability d , and an uninfected gamete X with probability $1 - d$; whereas an individual infected male produces infected gametes X^+ or Y^+ with probability $w/2$, and uninfected gametes X or Y with probability $(1 - w)/2$ each. Hence, the zygotes formed contribute to the offspring population. The Punnett squares of gametes of the basis elements are given in Table 2.

Table 2. Punnett squares of gametes from the mating of basis elements (a) XX vs. XY , (b) XX vs. XY^+ , (c) XX^+ vs. XY (d) XX^+ vs. XY^+ .

(a)

	X	Y
X	$\frac{1}{2}XX$	$\frac{1}{2}XY$

(b)

	X	X ⁺	Y	Y ⁺
X	$\frac{1}{2}(1-w)XX$	–	$\frac{1}{2}(1-w)XY$	–

(c)

	X	Y
X	$\frac{1}{2}(1-d)XX$	$\frac{1}{2}(1-d)XY$
X ⁺	$\frac{1}{2}dXX^+$	$\frac{1}{2}dXY^+$

(d)

	X	X ⁺	Y	Y ⁺
X	$\frac{1}{2}(1-d)(1-w)XX$	–	$\frac{1}{2}(1-d)(1-w)XY$	–
X ⁺	$\frac{1}{2}d(1-w)XX^+$	$\frac{1}{2}dwXX^+$	$\frac{1}{2}d(1-w)XY^+$	$\frac{1}{2}dwXY^+$

Summing up the information from the biological model in Table 2, the multiplication table of \mathcal{W} is achieved in Table 3.

Table 3. The multiplication table of \mathcal{W} .

	XY	XY ⁺
XX	$\frac{1}{2}XX + \frac{1}{2}XY$	$\frac{1}{2}(1-w)XX + \frac{1}{2}(1-w)XY$
XX ⁺	$\frac{1}{2}(1-d)XX + \frac{1}{2}dXX^+$ $+ \frac{1}{2}(1-d)XY + \frac{1}{2}dXY^+$	$\frac{1}{2}(1-d)(1-w)XX + \frac{1}{2}dXX^+$ $+ \frac{1}{2}(1-d)(1-w)XY + \frac{1}{2}dXY^+$

In \mathcal{W} , a type 1 female will never produce a type 2 male or female, regardless of whether she is crossed with a type 1 or type 2 male. Type 1 female crossing with type 2 male will produce type 1 female (male) with probability $\frac{1}{2}(1-w)$ (See Table 3).

Table 4. The inheritance coefficients $P_{ikj}^{(f)}$ of \mathcal{W} .

$P_{11,1}^{(f)} = 1$	$P_{12,1}^{(f)} = 1-w$	$P_{21,1}^{(f)} = 1-d$	$P_{22,1}^{(f)} = (1-d)(1-w)$
$P_{11,2}^{(f)} = 0$	$P_{12,2}^{(f)} = 0$	$P_{21,2}^{(f)} = d$	$P_{22,2}^{(f)} = d$

Also, let $P_{ikj}^{(f)}$ and $P_{ikj}^{(m)}$ be inheritance coefficients defined as the probability that an offspring (f for female, m for male) is type j , ($j = 1$ for non-infected and $j = 2$ for infected) when the parental pair is ik (i denotes the mother's and k denotes the father's types, respectively). Table 3 states the frequency of the individuals within the whole population, but to mimic the model in [7], multiply each coefficient by 2 in Table 3 to get the inheritance coefficients of females as the frequency of the female basis element to the total female population (Table 4).

Similarly, the inheritance coefficients of the male individuals are the same, so for all $i, j, k \in \{1, 2\}$, $P_{ikj}^{(f)} = P_{ikj}^{(m)}$. Notice that as there are deaths in the offspring of the mating of f_1 with m_2 and f_2 with m_2 , the sum of the inheritance coefficients in the second and fourth column respectively do not add up to 1, because w and d are greater than zero, less than or equal to one.

Now, the multiplication is defined on the basis \mathcal{B} as

$$f_i m_k = m_k f_i = \frac{1}{2} \left(P_{ik,1}^{(f)} f_1 + P_{ik,2}^{(f)} f_2 + P_{ik,1}^{(m)} m_1 + P_{ik,2}^{(m)} m_2 \right),$$

$$f_i f_j = 0, \quad i, j = 1, 2; \quad m_k m_l = 0, \quad k, l = 1, 2.$$

Hence,

$$\begin{aligned}
 f_1 m_1 &= \frac{1}{2} \left(P_{11,1}^{(f)} f_1 + P_{11,2}^{(f)} f_2 + P_{11,1}^{(m)} m_1 + P_{11,2}^{(m)} m_2 \right) \\
 &= \frac{1}{2} (f_1 + m_1) \\
 f_1 m_2 &= \frac{1}{2} \left(P_{12,1}^{(f)} f_1 + P_{12,2}^{(f)} f_2 + P_{12,1}^{(m)} m_1 + P_{12,2}^{(m)} m_2 \right) \\
 &= \frac{1}{2} (1 - w) (f_1 + m_1) \\
 f_2 m_1 &= \frac{1}{2} \left(P_{21,1}^{(f)} f_1 + P_{21,2}^{(f)} f_2 + P_{21,1}^{(m)} m_1 + P_{21,2}^{(m)} m_2 \right) \\
 &= \frac{1}{2} \left[(1 - d) (f_1 + m_1) + d (f_2 + m_2) \right] \\
 f_2 m_2 &= \frac{1}{2} \left(P_{22,1}^{(f)} f_1 + P_{22,2}^{(f)} f_2 + P_{22,1}^{(m)} m_1 + P_{22,2}^{(m)} m_2 \right) \\
 &= \frac{1}{2} \left[(1 - d)(1 - w) (f_1 + m_1) + d (f_2 + m_2) \right]
 \end{aligned} \tag{2}$$

Definition 3.1. The algebra \mathcal{W} generated by $\mathcal{B} = \{f_1, f_2, m_1, m_2\}$ for a given CI (paternal affection rate) w and maternal transmission rate d , with the multiplication in equation (2) is called *the evolution algebra of the Wolbachia-infected bisexual population*.

The population is described by its state vector $(x, y) \in S^1 \times S^1$, the product of two unit simplexes in \mathbb{R}^2 . Vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are the probability distributions of the females and males over the possible types which satisfy the equations:

$$x_i \geq 0, y_j \geq 0, \text{ for all } i, j \in \{1, 2\} \text{ and } \sum_{i=1}^2 x_i = 1 = \sum_{i=1}^2 y_i.$$

In terms of inheritance coefficients,

$$P_{ik,j}^{(f)} \geq 0, P_{ik,l}^{(m)} \geq 0, \sum_{j=1}^2 P_{i1,j}^{(f)} = 1 = \sum_{j=1}^2 P_{i1,j}^{(m)};$$

However,

$$\sum_{j=1}^2 P_{12,j}^{(f)} = 1 - w = \sum_{j=1}^2 P_{12,j}^{(m)} \text{ and } \sum_{j=1}^2 P_{22,j}^{(f)} = 1 - w + dw = \sum_{j=1}^2 P_{22,j}^{(m)} \tag{3}$$

Notice that, Wolbachia-infected bisexual population \mathcal{W} differs from EABP of [7] as listed:

- The first observation is that, unlike in an EABP, in \mathcal{W} when $k = 2$, as equation (3) states

$$\sum_{j=1}^2 P_{12,j}^{(f)} = 1 - w \neq 1 \quad \text{and} \quad \sum_{j=1}^2 P_{22,j}^{(f)} = 1 - w + dw \neq 1$$

- In [7, Remark 3.1], it states that if a population is free then the male and female types are identical and, in particular number of elements in the female basis is equal to the number of elements in the male basis (i.e. $n = v = 2$), the inheritance coefficients are the same for male and female offspring, that is

$$P_{ik,j} = P_{ik,j}^{(f)} = P_{ik,j}^{(m)}. \tag{4}$$

Although equation (4) is satisfied in \mathcal{W} , male and female types are not identical. For instance, whether crossed with a type 1 or type 2 male, a type 1 female will never produce a type 2 male or female. On

the other hand, a type 1 male crossed with type 2 female will produce an offspring of type 2 male or female with probability d (See Table 3). Hence, male and female types are not identical and \mathcal{W} is not a free population.

- Moreover, in \mathcal{W} , the symmetry condition $P_{ik,j} = P_{ki,j}$ is not necessarily satisfied, as $P_{12,1} = 1 - w \neq 1 - d = P_{21,1}$ and $P_{21,2} = d \neq 0 = P_{12,2}$.

4. Wolbachia-infected populations do not form a model for an evolution algebra

Consider \mathcal{W} with $\mathcal{B} = \{f_1, f_2, m_1, m_2\}$ for a given CI rate of w and maternal transmission rate d .

In a recent work of Bustamante, Mellon, and Velasco, the authors analyze when a genetic algebra is an evolution algebra by determining whether the structure matrices of the genetic algebra are simultaneously diagonalizable [2, Theorems 5,6]. Following the same notation, define $\pi_k : \mathcal{W} \rightarrow \mathbb{R}$ as $\pi_k(f_i f_j) = 0 = \pi_k(m_i m_j)$, $\pi_k(f_i m_j) = \frac{1}{2} P_{ij,k}$ for $k \in \{1, 2\}$. Equation (4), that is $P_{ij,k}^{(f)} = P_{ij,k} = P_{ij,k}^{(m)}$; reduces the four structural matrices to two different matrices.

By using Table 4, two distinct structural matrices are derived:

$$M_1(\mathcal{B}) = \begin{bmatrix} \pi_1(f_1 f_1) & \pi_1(f_1 f_2) & \pi_1(f_1 m_1) & \pi_1(f_1 m_2) \\ \pi_1(f_2 f_1) & \pi_1(f_2 f_2) & \pi_1(f_2 m_1) & \pi_1(f_2 m_2) \\ \pi_1(m_1 f_1) & \pi_1(m_1 f_2) & \pi_1(m_1 m_1) & \pi_1(m_1 m_2) \\ \pi_1(m_2 f_1) & \pi_1(m_2 f_2) & \pi_1(m_2 m_1) & \pi_1(m_2 m_2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1-w}{2} \\ 0 & 0 & \frac{1-d}{2} & \frac{(1-d)(1-w)}{2} \\ \frac{1}{2} & \frac{1-d}{2} & 0 & 0 \\ \frac{1-w}{2} & \frac{(1-d)(1-w)}{2} & 0 & 0 \end{bmatrix}$$

and

$$M_2(\mathcal{B}) = \begin{bmatrix} \pi_2(f_1 f_1) & \pi_2(f_1 f_2) & \pi_2(f_1 m_1) & \pi_2(f_1 m_2) \\ \pi_2(f_2 f_1) & \pi_2(f_2 f_2) & \pi_2(f_2 m_1) & \pi_2(f_2 m_2) \\ \pi_2(m_1 f_1) & \pi_2(m_1 f_2) & \pi_2(m_1 m_1) & \pi_2(m_1 m_2) \\ \pi_2(m_2 f_1) & \pi_2(m_2 f_2) & \pi_2(m_2 m_1) & \pi_2(m_2 m_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{d}{2} & \frac{d}{2} \\ 0 & \frac{d}{2} & 0 & 0 \\ 0 & \frac{d}{2} & 0 & 0 \end{bmatrix}.$$

Then $M_1(\mathcal{B})$ and $M_2(\mathcal{B})$ are symmetric matrices.

Theorem 4.1. \mathcal{W} is not an evolution algebra in the sense of Definition 2.5.

Proof. Recall the fact from [6, Theorem 1.3.12]: Assume A, B are $n \times n$ diagonalizable matrices. Then A, B are simultaneously diagonalizable if and only if A, B commute.

Now, $M_1(\mathcal{B}), M_2(\mathcal{B})$ are both diagonalizable with the diagonal matrices D_1 and D_2 :

$$D_1 = \begin{bmatrix} \frac{1}{2}\sqrt{(d^2 - 2d + 2)(w^2 - 2w + 2)} & & & 0 & & 0 & 0 \\ & 0 & & -\frac{1}{2}\sqrt{(d^2 - 2d + 2)(w^2 - 2w + 2)} & & 0 & 0 \\ & & 0 & & & 0 & 0 \\ & & & 0 & & 0 & 0 \\ & & & & 0 & & 0 & 0 \end{bmatrix}.$$

$$D_2 = \begin{bmatrix} \frac{d}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & -\frac{d}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ respectively. However,}$$

$$M_1(\mathcal{B})M_2(\mathcal{B}) = \begin{bmatrix} 0 & \frac{1}{4}d(2-w) & 0 & 0 \\ 0 & \frac{1}{4}d(1-d)(2-w) & 0 & 0 \\ 0 & 0 & \frac{1}{4}d(1-d) & \frac{1}{4}d(1-d) \\ 0 & 0 & \frac{1}{4}d(1-d)(1-w) & \frac{1}{4}d(1-d)(1-w) \end{bmatrix}$$

$$M_2(\mathcal{B})M_1(\mathcal{B}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{4}d(2-w) & \frac{1}{4}d(1-d)(2-w) & 0 & 0 \\ 0 & 0 & \frac{1}{4}d(1-d) & \frac{1}{4}d(1-d)(1-w) \\ 0 & 0 & \frac{1}{4}d(1-d) & \frac{1}{4}d(1-d)(1-w) \end{bmatrix}$$

That is, $M_1(\mathcal{B}), M_2(\mathcal{B})$ do not commute. Hence, by [6, Theorem 1.3.12] are not simultaneously diagonalizable. Therefore, \mathcal{W} is not an evolution algebra by [2]. \square

As stated in [7, Theorem 4.1] for an evolution algebra of a bisexual population, the following result is valid for an evolution algebra of a Wolbachia-infected bisexual population \mathcal{W} . Recall the definition of flexible and power-associative algebra.

Definition 4.2. An algebra A is called *flexible* if $z(tz) = (zt)z$ for any $z, t \in A$. An algebra A is *power-associative* if $(zz)(zz) = ((zz)z)z = (z(zz))z$ for every z of A .

Theorem 4.3. Let \mathcal{W} be the evolution algebra of a bisexual population with a Wolbachia infection.

- (1) \mathcal{W} is not necessarily associative.
- (2) \mathcal{W} is commutative and flexible.
- (3) \mathcal{W} is not necessarily power-associative.

Proof. (1) Take f_1, m_1 with $P_{11,1}^{(f)} = 1 \neq 0$ and take m_2 with $P_{12,1}^{(f)} = 1 - w \neq 0$ then

$$(f_1 m_1) m_2 = \frac{1}{2} (f_1 + m_1) m_2 = \frac{1}{2} f_1 m_2 = \frac{1}{4} (1 - w) (f_1 + m_1) \neq 0$$

But $f_1(m_1 m_2) = 0$, i.e. $(f_1 m_1) m_2 \neq f_1(m_1 m_2)$.

(2) It is clear that for any $z, t \in \mathbb{R}^{2 \times 2}$ we have

$$z = (x, y) = x_1 f_1 + x_2 f_2 + y_1 m_1 + y_2 m_2,$$

$$t = (u, v) = u_1 f_1 + u_2 f_2 + v_1 m_1 + v_2 m_2.$$

As $f_i m_k = m_k f_i$ ($i, k = 1, 2$), $zt = tz$.

Since \mathcal{W} is commutative, it follows that $(zt)z = z(tz)$ is true. Hence it is flexible.

- (3) To show that \mathcal{W} is not power-associative, we will construct an example of z such that $(zz)(zz) \neq ((zz)z)z$.

Consider $z = f_1 + m_2$. Then

$$z^2 = 2f_1m_2 = (1-w)(f_1 + m_1)$$

and

$$z^2z^2 = 2(1-w)^2f_1m_1 = (1-w)^2(f_1 + m_1).$$

On the other hand,

$$\begin{aligned} z^2z &= (1-w)(f_1m_1 + f_1m_2) \\ &= \frac{1}{2}(1-w)[(f_1 + m_1) + (1-w)(f_1 + m_1)] \\ &= \frac{1}{2}(1-w)(2-w)(f_1 + m_1) \end{aligned}$$

and

$$\begin{aligned} (z^2z)z &= \frac{1}{2}(1-w)(2-w)(f_1m_1 + f_1m_2) \\ &= \frac{1}{4}(1-w)(2-w)^2(f_1 + m_1). \end{aligned}$$

This shows that $(zz)(zz) \neq ((zz)z)z$. □

4.1. \mathcal{W} is not a baric algebra

Recall that an \mathbb{R} -algebra A is a baric algebra if it admits a nonzero algebra homomorphism $\sigma : A \rightarrow \mathbb{R}$.

Theorem 4.4. \mathcal{W} is not a baric algebra.

Proof. Consider a character $\sigma : \mathcal{W} \rightarrow \mathbb{R}$ such that $\sigma(f_i) = a_i, \sigma(m_j) = b_j$ for $i, j = 1, 2$. Now,

$$\begin{aligned} \sigma(f_i f_i) &= \sigma(f_i)\sigma(f_i) = a_i a_i \\ \sigma(0) &= 0 = a_i a_i \Rightarrow a_1 = a_2 = 0. \end{aligned}$$

In a similar manner, any basis vector should be mapped to 0. Hence, $\sigma \equiv 0$, \mathcal{W} does not have a nonzero character map. □

4.2. \mathcal{W} is a dibaric algebra when $w = 1 = d$

Theorem 4.5. \mathcal{W} is a dibaric algebra if and only if $w = 1 = d$.

Proof. Assume \mathcal{U} is the sex differentiation algebra generated by W and M . If \mathcal{W} is a dibaric algebra, then there is an onto homomorphism $\varphi : \mathcal{W} \rightarrow \mathcal{U}$.

Assume, $\varphi(f_i) = a_i W + a'_i M$ and $\varphi(m_j) = b_j W + b'_j M$ for $i, j = 1, 2$,

Then $\varphi(f_i f_i) = \varphi(0) = 0 = (a_i W + a'_i M)(a_i W + a'_i M) = (a_i a'_i)(W + M)$

implies $a_i a'_i = 0$, either $a_i = 0$ or $a'_i = 0$. A similar argument will show that either $b_j = 0$ and or $b'_j = 0$.

Claim: When the images of f_i are both nonzero, then either both f_1, f_2 are mapped to multiples of W or to multiples of M . A similar claim holds for m_j .

Assume on the contrary that $\varphi(f_1) = a_1 W$ and $\varphi(f_2) = a'_2 M$. Then $\varphi(f_1 f_2) = \varphi(0) = 0 = (a_1 W)(a'_2 M) = \frac{1}{2}(a_1 a'_2)(W + M)$ and so $a_1 a'_2 = 0$. Hence, $a_1 = 0$ or $a'_2 = 0$. This is a contradiction to the assumption of the claim.

Table 5. 18 possible onto homomorphisms φ .

$\varphi(f_1)$	a'_1M	a'_1M	0	a'_1M	a'_1M	0	a'_1M	a'_1M	0
$\varphi(f_2)$	0	a'_2M	a'_2M	0	a'_2M	a'_2M	0	a'_2M	a'_2M
$\varphi(m_1)$	b_1W	b_1W	b_1W	b_1W	b_1W	0	0	0	0
$\varphi(m_2)$	0	0	0	b_2W	b_2W	b_2W	b_2W	b_2W	b_2W
$\varphi(f_1)$	a_1W	a_1W	0	a_1W	a_1W	0	a_1W	a_1W	0
$\varphi(f_2)$	0	a_2W	a_2W	0	a_2W	a_2W	0	a_2W	a_2W
$\varphi(m_1)$	b'_1M	b'_1M	b'_1M	b'_1M	b'_1M	b'_1M	0	0	0
$\varphi(m_2)$	b'_2M	b'_2M	b'_2M	0	0	0	b'_2M	b'_2M	b'_2M

The images of f_j are either both nonzero and mapped to multiples of W or both nonzero and mapped to multiples of M . Thus, there are 18 possible maps as shown in Table 5.

Without loss of generality, take φ as, for $i, j = 1, 2$, $\varphi(f_i) = a_iW$, $\varphi(m_j) = b_jM$ where $a_i, b_j \in \mathbb{R}$. For

$$\begin{aligned} z &= (x, y) = x_1f_1 + x_2f_2 + y_1m_1 + y_2m_2, \\ t &= (u, v) = u_1f_1 + u_2f_2 + v_1m_1 + v_2m_2 \end{aligned}$$

we obtain

$$\begin{aligned} zt &= u_1y_1f_1m_1 + v_1x_1f_1m_1 + u_1y_2f_1m_2 + u_2y_1f_2m_1 \\ &\quad + x_2v_1f_2m_1 + x_1v_2f_1m_2 + u_2y_2f_2m_2 + x_2v_2f_2m_2 \\ &= (x_1v_1 + u_1y_1)\frac{1}{2}(f_1 + m_1) + (x_1v_2 + u_1y_2)\frac{1}{2}(1-w)(f_1 + m_1) \\ &\quad + (x_2v_1 + u_2y_1)\frac{1}{2}[(1-d)(f_1 + m_1) + d(f_2 + m_2)] \\ &\quad + (x_2v_2 + u_2y_2)\frac{1}{2}[(1-d)(1-w)(f_1 + m_1) + d(f_2 + m_2)] \\ zt &= \frac{1}{2} \left\{ \left[(x_1v_1 + u_1y_1) + (x_1v_2 + u_1y_2)(1-w) + (x_2v_1 + u_2y_1)(1-d) \right. \right. \\ &\quad \left. \left. + (x_2v_2 + u_2y_2)(1-d)(1-w) \right] (f_1 + m_1) \right. \\ &\quad \left. + (x_2v_1 + u_2y_1 + x_2v_2 + u_2y_2)d(f_2 + m_2) \right\} \\ \varphi(zt) &= \frac{1}{2} \left\{ \left[(x_1v_1 + u_1y_1) + (x_1v_2 + u_1y_2)(1-w) + (x_2v_1 + u_2y_1)(1-d) \right. \right. \\ &\quad \left. \left. + (x_2v_2 + u_2y_2)(1-d)(1-w) \right] (a_1W + b_1M) \right. \\ &\quad \left. + (x_2v_1 + u_2y_1 + x_2v_2 + u_2y_2)d(a_2W + b_2M) \right\} \end{aligned}$$

On the other hand

$$\begin{aligned} \varphi(z)\varphi(t) &= [(x_1a_1 + x_2a_2)W + (y_1b_1 + y_2b_2)M][(u_1a_1 + u_2a_2)W + (v_1b_1 + v_2b_2)M] \\ &= \frac{1}{2} \{ a_1b_1(x_1v_1 + u_1y_1) + a_1b_2(x_1v_2 + u_1y_2) \\ &\quad + a_2b_1(x_2v_1 + u_2y_1) + a_2b_2(x_2v_2 + u_2y_2) \} (W + M) \end{aligned}$$

Since φ is a homomorphism $\varphi(zt) = \varphi(z)\varphi(t)$ and comparing coefficients of W and M we get

$$a_1 = a_1b_1 \text{ and } b_1 = a_1b_1 \quad (5)$$

$$(1-d)a_1 + da_2 = a_2b_1 \text{ and } (1-d)b_1 + db_2 = a_2b_1 \quad (6)$$

$$(1-w)a_1 = a_1b_2 \text{ and } (1-w)b_1 = a_1b_2 \quad (7)$$

$$(1-d)(1-w)a_1 + da_2 = a_2b_2 \text{ and } (1-d)(1-w)b_1 + db_2 = a_2b_2 \quad (8)$$

By equation (5), $a_1 = b_1 = 0$ or $a_1 = b_1 = 1$.

Case 1: If $a_1 = b_1 = 0$, then by equation (6), either $d = 0$ or $a_2 = 0 = b_2$. However, if the maternal transmission rate is zero $d = 0$, then the offspring population will not be infected anymore, which is not biologically meaningful. Hence, $a_2 = b_2 = 0$ and φ is the zero homomorphism. Contradiction to the assumption that \mathcal{W} is dibaric.

Case 2: If $a_1 = b_1 = 1$, then by equation (6), either $d = 0$ or $a_2 = b_2$. As $d = 0$ is not possible, take $a_2 = b_2$. By equation (7), $a_2 = b_2 = 1 - w$. Then from equation (8), either $w = 0$ or $w = 1$. Again, $w = 0$ is not considered, so $a_2 = b_2 = 0$. Now, plug the values into equation (6), to get $d - 1 = 0$. Hence, $d = 1$ also. That is, there is a nonzero homomorphism φ from \mathcal{W} onto \mathcal{U} , such that $\varphi(f_1) = W$, $\varphi(m_1) = M$, $\varphi(f_2) = 0$, $\varphi(m_2) = 0$. (The other possible nonzero homomorphism φ maps the basis vectors as follows: $\varphi(f_1) = M$, $\varphi(m_1) = W$, $\varphi(f_2) = 0$, $\varphi(m_2) = 0$.)

Therefore, if the algebra \mathcal{W} is dibaric then $w = 1$ and $d = 1$.

Conversely, assume $w = 1 = d$, and show that \mathcal{W} , whose multiplication is given in Table 6, is a dibaric algebra.

Table 6. The multiplication table of \mathcal{W} when $w = 1 = d$.

	XY	XY^+
XX	$\frac{1}{2}XX + \frac{1}{2}XY$	—
XX^+	$\frac{1}{2}XX^+ + \frac{1}{2}XY^+$	$\frac{1}{2}XX^+ + \frac{1}{2}XY^+$

Define $\varphi : \mathcal{W} \rightarrow \mathcal{U}$ as $\varphi(f_1) = W$, $\varphi(m_1) = M$, $\varphi(f_2) = 0$, $\varphi(m_2) = 0$. A straightforward computation reveals that φ is an onto homomorphism. \square

4.3. \mathcal{W} contains the sex differentiation algebra as a subalgebra

As Table 3 reveals, the evolution algebra of a Wolbachia-infected bisexual population \mathcal{W} contains a sex differentiation subalgebra $\mathcal{U} = \langle f_1, m_1 \rangle = \langle XX, XY \rangle$. However, $\mathcal{U} \cdot \mathcal{W} \not\subseteq \mathcal{U}$, so the subalgebra \mathcal{U} is not an ideal of \mathcal{W} .

Sex differentiation algebra \mathcal{U} is a dibaric algebra by definition. Moreover, $\mathcal{U}^2 = \langle f_1 + m_1 \rangle$ is a baric algebra which is an ideal of \mathcal{U} and a subalgebra of \mathcal{W} .

5. Fixed points of Wolbachia-infected populations

This problem is also considered in the first paper of the sequel *An Algebraic Discussion of Bisexual Populations with Wolbachia Infection I: Discrete Dynamical System Approach*, [8] via the discrete dynamical system approach. The fixed points are calculated and classified as stable and non-stable. Here, we use the evolutionary operator to arrive to the same conclusion.

Consider the map $V : S^1 \times S^1 \rightarrow S^1 \times S^1$ where

$$S^1 \times S^1 = \left\{ (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2, x_i, y_i \geq 0, \sum_{i=1}^2 x_i = 1 = \sum_{i=1}^2 y_i \right\}$$

with $z^0 = (x_1^0, x_2^0, y_1^0, y_2^0) \mapsto z' = (x_1', x_2', y_1', y_2')$.

In the evolution algebra of a bisexual population (EABP) set-up, inheritance coefficients satisfy the equation (1) and $z' = (x'_1, x'_2, y'_1, y'_2)$ is defined as

$$x'_j = \sum_{i,k=1}^2 P_{ik,j}^{(f)} x_i y_k \quad y'_j = \sum_{i,k=1}^2 P_{ik,j}^{(m)} x_i y_k \quad \text{for } j = 1, 2.$$

However, in \mathcal{W} , there are deaths in the offspring, hence equation (1) is not satisfied. To overcome this flaw, normalize the offspring vector with respect to the ℓ^1 -norm. Define

$$x'_j = \frac{\sum_{i,k=1}^2 P_{ik,j}^{(f)} x_i y_k}{\sum_{i,j,k=1}^2 P_{ik,j}^{(f)} x_i y_k}, \quad y'_j = \frac{\sum_{i,k=1}^2 P_{ik,j}^{(m)} x_i y_k}{\sum_{i,j,k=1}^2 P_{ik,j}^{(m)} x_i y_k} \quad \text{for } j = 1, 2. \tag{9}$$

By using Table 4, a direct computation shows that

$$\sum_{i,j,k=1}^2 P_{ik,j}^{(f)} x_i y_k = 1 - wy_2 + dwx_2y_2 = \sum_{i,j,k=1}^2 P_{ik,j}^{(m)} x_i y_k.$$

This map V is the evolutionary operator that describes the inheritance process of \mathcal{W} from one generation (initial state vector) to the offspring (next generation's state vector). Namely, $V(z) = \frac{z \cdot z}{\|z \cdot z\|_1}$ where $\|z \cdot z\|_1 = 1 - wy_2 + dwx_2y_2$.

Two important biological questions arise:

Q1) For what value of z , $V(z) = z$? (The population stabilizes at this state vector.)

Q2) For what value of z , $V(z) = 0$? (The population dies if reaches this state vector.)

The element z that solves Q1) is an idempotent, the element z that solves Q2) is an absolute nilpotent element of the algebra \mathcal{W} . As both d, w are parameters of \mathcal{W} , the solution should depend on the CI paternal affection rate w and maternal transmission rate d .

5.1. Idempotent elements of \mathcal{W}

An idempotent element of \mathcal{W} is a fixed point of the operator V . Let $z = (x, y) = x_1f_1 + x_2f_2 + y_1m_1 + y_2m_2$, be a fixed point of V . Then $V(z) = \frac{z \cdot z}{\|z \cdot z\|_1} = z$, that is $z \cdot z = (\|z \cdot z\|_1) z = (\sum_{i,j,k=1}^2 P_{ik,j} x_i y_k) z = (1 - wy_2 + dwx_2y_2)z$. Now,

$$z \cdot z = [x_1y_1 + x_1y_2(1 - w) + x_2y_1(1 - d) + x_2y_2(1 - d)(1 - w)](f_1 + m_1) + [(x_2y_1 + x_2y_2)d](f_2 + m_2) \tag{*}$$

and

$$\left(\sum_{i,j,k=1}^2 P_{ik,j} x_i y_k\right) z = (1 - wy_2 + dwx_2y_2)(x_1f_1 + x_2f_2 + y_1m_1 + y_2m_2). \tag{**}$$

Note that the coefficients of f_i and m_i are equal in $z \cdot z$ (See (*)), so $x_1 = y_1$ and $x_2 = y_2$.

(Notice that this result agrees with the [8, Proposition 3.1], even if the frequencies of initial Wolbachia infected male and female populations differ, in the next generation f_2 and m_2 frequencies become equal, furthermore, they are equal in the fixed point.) Also, use the fact that $x_1 + x_2 = 1$ and simplify both (*) and (**) using x_2 as the parameter, then we have the following equalities, respectively:

$$z \cdot z = [(1 - x_2)^2 + (1 - x_2)x_2(2 - w - d) + x_2^2(1 - d)(1 - w)](f_1 + m_1) + dx_2(f_2 + m_2)$$

and

$$(1 - wx_2 + dwx_2^2)z = (1 - wx_2 + dwx_2^2)(1 - x_2)(f_1 + m_1) + (1 - wx_2 + dwx_2^2)x_2(f_2 + m_2).$$

As the equations in (*) and (**) are equal, the coefficients of the basis vectors are the same in both equations. The coefficient of f_1 is

$$(1 - wx_2 + dwx_2^2)(1 - x_2) = (1 - x_2)^2 + (1 - x_2)x_2(2 - w - d) + x_2^2(1 - d)(1 - w)$$

and the coefficient of f_2 is

$$(1 - wx_2 + dwx_2^2)x_2 = dx_2.$$

Simplification produces the same equation in both equalities:

$$x_2(dw(x_2)^2 - wx_2 + (1 - d)) = 0.$$

If $x_2 = 0$, then $x_1 = 1$, hence the population is at a trivial fixed point, and there is no Wolbachia infection in the population. In this case, $f_1 + m_1$ is an idempotent element.

Also, there are two solutions to $dw(x_2)^2 - wx_2 + (1 - d) = 0$, namely

$$x_2 = \frac{1}{2d} \left(1 + \frac{\sqrt{w(4d^2 - 4d + w)}}{w} \right) \text{ and } x_2 = \frac{1}{2d} \left(1 - \frac{\sqrt{w(4d^2 - 4d + w)}}{w} \right) \text{ provided that } 0 < d(1 - d) < w/4.$$

On the other hand, if $d = 1$.

$$x_2 = \frac{1}{2} \left(1 \pm \frac{\sqrt{w(4 - 4 + w)}}{w} \right) \text{ implies } x_2 = 0 \text{ or } x_2 = 1. \text{ Summarizing,}$$

$$x_2 \in \begin{cases} \{0, 1\} & \text{if } d = 1 \\ \left\{ \frac{1}{2d} \left(1 \pm \frac{\sqrt{w(4d^2 - 4d + w)}}{w} \right) \right\} & \text{if } 0 < d(1 - d) < w/4 \end{cases}$$

Hence, the idempotent element z will be:

Case 1. If $d = 1$, then either

a) $x_2 = 0$ implies $x_1 = 1$. Then $z = f_1 + m_1$.

b) $x_2 = 1$ implies $x_1 = 0$. Then $z = f_2 + m_2$,

To verify, a)

$$z \cdot z = (f_1 + m_1)(f_1 + m_1) = 2 \frac{1}{2} (f_1 + m_1) = f_1 + m_1 = z$$

Here, $\|z \cdot z\|_1 = 1 - wy_2 + dwx_2y_2 = 1 - 0 - 0 = 1$.

b)

$$z \cdot z = (f_2 + m_2)(f_2 + m_2) = (1 - w)(1 - d)(f_1 + m_1) + d(f_2 + m_2) = f_2 + m_2 = z$$

Here, $\|z \cdot z\|_1 = 1 - wy_2 + dwx_2y_2 = 1 - w + w = 1$.

Case 2. If $0 < d(1 - d) < w/4$,

$$x_2 = \frac{1}{2d} \left(1 \pm \frac{\sqrt{w(4d^2 - 4d + w)}}{w} \right) \text{ implies } x_1 = 1 - \frac{1}{2d} \left(1 \pm \frac{\sqrt{w(4d^2 - 4d + w)}}{w} \right). \text{ Then}$$

$$z = \left(1 - \frac{1}{2d} \mp \frac{\sqrt{w(4d^2 - 4d + w)}}{2dw} \right) (f_1 + m_1) + \left(\frac{1}{2d} \pm \frac{\sqrt{w(4d^2 - 4d + w)}}{2dw} \right) (f_2 + m_2)$$

To verify:

Here, a computation reveals that

$$\begin{aligned} \|z \cdot z\|_1 &= 1 - wy_2 + dwx_2y_2 = 1 - wx_2 + dw(x_2)^2 \\ &= 1 - \frac{w}{2d} \left(1 \pm \frac{\sqrt{w(4d^2 - 4d + w)}}{w} \right) + \frac{w}{4d} \left(1 \pm \frac{\sqrt{w(4d^2 - 4d + w)}}{w} \right)^2 = d. \end{aligned}$$

$$z \cdot z = \left(\frac{2dw - w \mp \sqrt{w(4d^2 - 4d + w)}}{2w} \right) (f_1 + m_1) + \frac{1}{2d} \left(1 \pm \frac{\sqrt{w(4d^2 - 4d + w)}}{w} \right) d(f_2 + m_2)$$

Note that, $\frac{z \cdot z}{\|z \cdot z\|_1} = \frac{1}{d}(z \cdot z) = z$ as expected.

Summarizing the results, the set of idempotent elements of \mathcal{W} , denoted by $Idem(\mathcal{W})$ is given in [Theorem 5.1](#).

Theorem 5.1. When $d = 1$, $Idem(\mathcal{W}) = \{f_1 + m_1, f_2 + m_2\}$.

When $0 < d(1 - d) \leq w/4$,

$$Idem(\mathcal{W}) = \left\{ \left(1 - \frac{1}{2d} \mp \frac{\sqrt{w(4d^2 - 4d + w)}}{2dw} \right) (f_1 + m_1) + \left(\frac{1}{2d} \pm \frac{\sqrt{w(4d^2 - 4d + w)}}{2dw} \right) (f_2 + m_2) \right\}$$

[Theorem 5.1](#) agrees with the fixed points calculated via the dynamical system approach in [8, Section 3]. For instance,

1. When $d = w = \frac{3}{4}$, the fixed point is $z = \frac{1}{3}(f_1 + m_1) + \frac{2}{3}(f_2 + m_2)$ [8, Case 2 of Section 3].
2. When $w = 1$, that is CI expression is 100%, either,

$$x_2 = \frac{1}{2d} + \frac{\sqrt{(4d^2 - 4d + 1)}}{2d} = \frac{1 + 2d - 1}{2d} = 1$$

The idempotent element is $z = 0(f_1 + m_1) + 1(f_2 + m_2)$.

or the idempotent depends on $d \in (0, 1]$ with

$$x_2 = \frac{1}{2d} - \frac{\sqrt{(4d^2 - 4d + 1)}}{2d} = \frac{1 - 2d + 1}{2d} = \frac{1 - d}{d}$$

provided that $\frac{1-d}{d} \leq 1$, (i.e. $\frac{1}{2} \leq d$). The idempotent element is $z = \frac{2d-1}{d}(f_1 + m_1) + \frac{1-d}{d}(f_2 + m_2)$.

Also, take $w = 1, d = 1/2$. Then $z = f_2 + m_2$ is the only idempotent element.

Hence, for $w = 1$, if the maternal transmission rate d is greater than 50%, there are two fixed points of the population. Otherwise, when $0 < d \leq \frac{1}{2}$, the only idempotent is $(f_2 + m_2)$.

Consider a particular scenario: take a population with $w = 1, d = \frac{2}{3}$. The inequality $d(1 - d) = \frac{2}{9} \leq \frac{w}{4} = \frac{1}{4}$ is satisfied and $x_2 = \frac{1-d}{d} = \frac{1}{2} \leq 1$. Then $z = \frac{1}{2}(f_1 + m_1) + \frac{1}{2}(f_2 + m_2)$ is an idempotent.

5.2. Absolute nilpotent elements of \mathcal{W}

The element z is called *absolute nilpotent* if $z^2 = 0$.

Let $z = (x, y) = x_1f_1 + x_2f_2 + y_1m_1 + y_2m_2$ be an absolute nilpotent element of \mathcal{W} , that is $V(z) =$

$$\frac{1}{\|z\|_1} (z \cdot z) = 0 \text{ where } \|z\|_1 = \sum_{i,j,k=1}^2 P_{ik,j} x_i y_k = 1 - wy_2 + dwx_2y_2.$$

Hence,

$$0 = z \cdot z = [x_1y_1 + x_1y_2(1-w) + x_2y_1(1-d) + x_2y_2(1-d)(1-w)](f_1 + m_1) \\ + (x_2y_1 + x_2y_2)d(f_2 + m_2).$$

Consequently,

$$(I) \quad x_1y_1 + x_1y_2(1-w) + x_2y_1(1-d) + x_2y_2(1-d)(1-w) = 0$$

and

$$(II) \quad (x_2y_1 + x_2y_2)d = 0.$$

Take (II), since $y_1 + y_2 = 1$, and $d \neq 0$, $x_2 = 0$, and $x_1 = 1$. Substituting $x_2 = 0$ and $x_1 = 1$ in (I), gives $y_1 + y_2(1-w) = 0$. Then, $1 - wy_2 = 0$ implies $y_2 = \frac{1}{w}$. Note that $w \leq 1$ implies $1/w \geq 1$ and $y_i \leq 1$ for $i = 1, 2$. Thus, $y_2 = \frac{1}{w} = 1$ occurs when $w = 1$. Consequently, $y_1 = 0$, and $z = f_1 + m_2$ is an absolute nilpotent element of \mathcal{W} with $w = 1$. The set of all absolute nilpotent elements of \mathcal{W} with $w = 1$ will be the subalgebra generated by $\langle f_1 + m_2 \rangle$. If $w \neq 1$, there are no nonzero absolute nilpotent elements, i.e. the following theorem is proved:

Theorem 5.2. For \mathcal{W} with \mathcal{CI} given as w and maternal transmission rate d , the set of absolute nilpotent elements $Nil(\mathcal{W})$ is zero except when $w = 1$. That is, for any value of $d \in (0, 1]$,

$$Nil(\mathcal{W}) = \begin{cases} \langle f_1 + m_2 \rangle & \text{if } w = 1 \\ \{0\} & \text{if } w \neq 1. \end{cases}$$

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Disclosure statement

The authors report that there are no competing interests to declare.

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