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Hankel Determinants of Normalized Analytic Functions Associated with Hyperbolic Secant Function

Sushil Kumar ¹, Daniel Breaz ², Luminita-Ioana Cotîrlă ³  and Asena Çetinkaya ^{4,*} ¹ Bharati Vidyapeeth's College of Engineering, Delhi 110063, India; sushilkumar16n@gmail.com² Department of Mathematics, "1 Decembrie 1918" University of Alba-Iulia, 510009 Alba-Iulia, Romania; dbreaz@uab.ro³ Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania; luminita.cotirla@math.utcluj.ro⁴ Department of Mathematics and Computer Science, İstanbul Kültür University, 34158 İstanbul, Türkiye

* Correspondence: asnfigen@hotmail.com

Abstract: In this paper, we consider a subclass of normalized analytic functions associated with the hyperbolic secant function. We compute the sharp bounds on third- and fourth-order Hermitian–Toeplitz determinants for functions in this class. Moreover, we determine the bounds on second- and third-order Hankel determinants, as well as on the generalized Zalcman conjecture. We examine a Briot–Bouquet-type differential subordination involving the Bernardi integral operator. Finally, we obtain a univalent solution to the Briot–Bouquet differential equation, and discuss the majorization property for such function classes.

Keywords: Briot–Bouquet differential subordination; Hermitian–Toeplitz and Hankel determinants; hyperbolic secant function; starlike functions; majorization

MSC: 30C45; 30C50



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1. Introduction and Preliminaries

The hyperbolic secant function $\operatorname{sech}(\zeta)$ is an entire univalent function as well as a periodic function with respect to the imaginary part of ζ . The function $\operatorname{sech}(\zeta)$ is used in various fields of mathematics and science to describe the shape of certain curves. It occurs in the solutions of differential equations, cubic equations, quantum mechanics, the design of digital filters, and in the applications of signal processing. For instance, an analytic expression for a notch filter was established by using the hyperbolic secant function, and the proposed filter was compared with some common windows used in signal processing [1]. The hyperbolic secant function also occurs in statistics and probability theory, where it is used to define the hyperbolic secant distribution, which is a continuous probability distribution. In [2], the alpha-skew hyperbolic secant distribution was demonstrated, and the cumulative distribution function, non-central moments, skewness, kurtosis, moment-generating function, and characteristic function were determined for this new skewed distribution, defined by the hyperbolic secant distribution. For more details on the hyperbolic secant function and its applications, we refer readers to [3] and the references therein.

There are various geometric properties of the analytic functions, such as convexity, starlikeness, univalence, close-to-convexity, subordination inclusions and the bounds of coefficients' functionals. These geometric properties are useful for analyzing the behavior of analytic functions; therefore, several applications of such functions in various areas can be determined. In this sequel, it is observed that the Hermitian–Toeplitz determinants play a central role in the study of random matrices, which are used to model complex systems in physics, statistics, and other fields. The Hankel determinant is also a special type of

determinant that arises from a matrix whose elements are arranged in a Hankel pattern, meaning that the matrix is constructed by taking a sequence of numbers and placing them in diagonal lines such that each diagonal has constant values. The Hankel determinant is often used to study and analyze the Fibonacci sequence or the Lucas sequence. It can be used to calculate the generating function of a sequence and its behavior as the sequence grows. It appears in the theory of orthogonal polynomials, which are polynomials that are orthogonal with respect to some weight function. The Hermitian–Toeplitz determinant of q^{th} order is given by $T_q(n) := [a_{ij}]$ such that $a_{ij} = a_{n+j-i}$ for $j \geq i$ and $a_{ij} = \overline{a_{ji}}$ for $j < i$, where a_{ij} are initial coefficients of the analytic functions. In a similar way, the Hankel determinant of the q^{th} order is given as $H_q(n) := \det\{a_{n+i+j-2}\}_{i,j}^q$, where $1 \leq i, j \leq q$, $a_1 = 1$. Hence,

$$T_3(1) := 2 \operatorname{Re}(a_2^2 \overline{a_3}) - 2|a_2|^2 - |a_3|^2 + 1, \quad (1)$$

$$T_4(1) := 1 - 2\operatorname{Re}(a_2^3 \overline{a_4}) + 4\operatorname{Re}(a_2^2 \overline{a_3}) - 2\operatorname{Re}(a_2 \overline{a_3}^2 a_4) + 4\operatorname{Re}(a_2 a_3 \overline{a_4}) + |a_2|^4 - 3|a_2|^2 + |a_3|^4 - 2|a_3|^2 + |a_2|^2 |a_4|^2 - 2|a_2|^2 |a_3|^2 - |a_4|^2, \quad (2)$$

and

$$H_2(3) := a_3 a_5 - a_4^2, \quad (3)$$

$$H_3(1) := -a_2^2 a_5 + 2a_2 a_3 a_4 - a_3^3 + a_3 a_5 - a_4^2, \quad (4)$$

$$H_3(2) := a_2 a_4 a_6 - a_2 a_5^2 - a_3^2 a_6 + 2a_3 a_4 a_5 - a_4^3. \quad (5)$$

In 1975, Louis Zalcman proposed the Zalcman conjecture, which suggests a universal bound on the growth of certain sequences associated with univalent functions. This conjecture also addresses the growth behavior of certain sequences associated with univalent functions, and provides an estimate of $a_n^2 - a_{2n-1}$ for $n \geq 2$. For more details, refer to [4–6].

Let $\mathcal{A}$ denote the class of all analytic functions f of the form $f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n$ defined in the open unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and let $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of all univalent functions. The analytic function f_1 is subordinate to the analytic function f_2 , written as $f_1 \prec f_2$, if there exists a Schwarz function ω which is analytic on $\mathbb{D}$ such that $f_1 = f_2 \circ \omega$ (see [7]). A function $f \in \mathcal{A}$ is starlike if $f(\mathbb{D})$ is a starlike region with respect to the origin. In the last three decades, many authors introduced various subclasses of starlike functions related to the bounded symmetric regions lying in the right half-plane, and investigated their many geometric properties, such as their radius estimates, differential subordination inclusions, coefficients, inequalities, and majorization properties (see, for example [8,9]). In this sequel, recently, Al-Shbeil et al. [10] considered a subclass $\mathcal{S}_{\text{sech}}^*$ of starlike functions related to the hyperbolic secant function $\operatorname{sech}(\zeta)$. If a function f is in $\mathcal{S}_{\text{sech}}^*$, then the subordination relation $\zeta f'(\zeta)/f(\zeta) \prec \operatorname{sech}(\zeta) = 2/(e^\zeta + e^{-\zeta})$, ($\zeta \in \mathbb{D}$) holds. The functions $\zeta e^{\zeta/3}$ and $\zeta + \zeta^2/4$ are examples of $\mathcal{S}_{\text{sech}}^*$. These authors examined several differential subordination results related to the Janowski function by using a method used by Jack (see [11]). Further, Bano et al. [12] determined the structural formula, sharp radius of starlikeness, and radius of convexity for functions $f \in \mathcal{S}_{\text{sech}}^*$.

The Briot–Bouquet subordination is a special case of more general differential subordination theory, which is applied in studies of the various properties of complex-valued functions such as univalence, convexity, and starlikeness. The Briot–Bouquet subordination is given by

$$\phi(\zeta) + \frac{\zeta \phi'(\zeta)}{\gamma \phi(\zeta) + \mu} \prec \kappa(\zeta), \quad (\gamma, \mu \in \mathbb{C}; \gamma \neq 0) \quad (6)$$

with $\phi(0) = \kappa(0) = 1$. If $q(\zeta) = 1 + q_1 \zeta + q_2 \zeta^2 + \dots$ has the property that $\phi \prec q$ for any function ϕ , satisfying the condition given in (6), then it is said to be a dominant of (6). For more details, we refer to [13].

Understanding majorization in the context of analytic functions is valuable for analyzing function spaces, studying functions behavior, and establishing relationships between

different classes of functions in complex analysis. For instance, it is used to study the behavior of conformal mappings between two regions in the complex plane. Through majorization property, we study convexity and starlikeness of analytic functions, and examine the behavior of functions on the boundary of the unit disk. The analytic function f is majorized to the analytic function g , denoted by $f(\zeta) \ll g(\zeta)$ ($\zeta \in \mathbb{D}$) [14], if there exists an analytic function ϕ in $\mathbb{D}$ such that

$$|\phi(\zeta)| \leq 1 \text{ and } f(\zeta) = \phi(\zeta)g(\zeta), \quad (\zeta \in \mathbb{D}).$$

In [15], sharp estimates of the Hermitian–Toeplitz determinant for some analytic functions were determined. Further, Jastrzębski et al. [16] discussed estimates of $T_3(1)$ for close-to-star functions. In [17], Lecko et al. computed sharp estimates of $T_4(1)$ for convex functions. The first estimates of the Hankel determinants for functions $f \in \mathcal{S}$ were discussed in [18,19]. Sim et al. [20] computed the sharp estimates of a Hankel determinant of the second order for the classes of strongly starlike and strongly convex functions of order β . Babalola [21] first discussed the bounds of $H_3(1)$ for starlike and convex functions. Later, the estimate of $H_3(1)$ for starlike functions of order $1/2$ were discussed in [22]. In [23], the authors computed the sharp lower and upper bounds for the third-order Hermitian–Toeplitz determinant for functions with a bounded turning of order α . Obradovic and Tuneski [24] determined the sharp bound of the third order Hermitian–Toeplitz determinant for univalent functions. Further, in [25], the authors computed sharp bounds for the second-order Hankel determinant, the Zalcman functional, and some Hermitian–Toeplitz determinants of Ozaki’s close-to-convex functions. Ma [26] estimated the Zalcman conjecture for a subclass of convex functions. Further, the generalized Zalcman conjecture $a_m a_n - a_{m+n-1}$ was studied, where $m \geq 2, n \geq 2$ was explored for starlike functions [27].

In this paper, we first determine the sharp bounds of the Hermitian–Toeplitz determinants $T_3(1)$ and $T_4(1)$ for the functions $f \in \mathcal{S}_{\text{sech}}^*$. We compute a bound of the Hankel determinant $H_2(3)$ involving the initial coefficients and initial inverse coefficients. Further, we compute the bounds of the Hankel determinants $H_3(1)$ and $H_3(2)$ involving initial coefficients as well as the bound of the generalized Zalcman conjecture $|a_5 - a_3^2|$. By using the Bernardi integral operator, we establish a Briot–Bouquet differential subordination. Also, we obtain a univalent solution to the Briot–Bouquet differential equation for functions $f \in \mathcal{S}_{\text{sech}}^*$. Finally, we investigate the majorization property for such function classes.

To prove the main results, we need the following lemmas.

Lemma 1 ([28], Lemma 3, p. 254). *Let $\mathcal{P}$ be the class of analytic functions*

$$p(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + \cdots$$

satisfying $\operatorname{Re}(p(\zeta)) > 0$ ($\zeta \in \mathbb{D}$). Then, $2p_2 = p_1^2 + (4 - p_1^2)\xi$ for some $\xi \in \overline{\mathbb{D}}$.

Lemma 2 ([29], Lemma 2.3, p. 507). *Let $p \in \mathcal{P}$. Then, for all $n, m \in \mathbb{N}$,*

$$|vp_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq v \leq 1; \\ 2|2v - 1|, & \text{elsewhere.} \end{cases}$$

If $0 < v < 1$, then the inequality is sharp for $p(\zeta) = (1 + \zeta^{m+n})/(1 - \zeta^{m+n})$. In other cases, the inequality is sharp for $p_0(\zeta) = (1 + \zeta)/(1 - \zeta)$.

Lemma 3 ([30]). *Let $p \in \mathcal{P}$. Then, for any real number v ,*

$$|vp_3 - p_1^3| \leq \begin{cases} 2|v - 4|, & v \leq \frac{4}{3}; \\ 2v\sqrt{\frac{v}{v-1}}, & v > \frac{4}{3}. \end{cases}$$

If $\nu \leq \frac{4}{3}$, equality holds for $p_0(\zeta) := (1 + \zeta)/(1 - \zeta)$, and if $\nu > \frac{4}{3}$, then equality holds for

$$p_1(\zeta) := \frac{1 - \zeta^2}{\zeta^2 - 2\sqrt{\frac{\nu}{\nu-1}}\zeta + 1}.$$

Lemma 4 ([31]). Let ω be a Schwarz function of the form $\omega(\zeta) = c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \dots$. Then,

$$|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2, \quad |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}, \quad |c_4| \leq 1 - |c_1|^2 - |c_2|^2.$$

2. Main Results

In this section, we first examine sharp estimates of the third- and the fourth-order Hermitian-Toeplitz determinants for the functions $f \in \mathcal{S}_{\text{sech}}^*$.

Theorem 1. Let $f \in \mathcal{A}$ be in the class $\mathcal{S}_{\text{sech}}^*$. Then,

$$\frac{15}{16} \leq T_3(1) \leq 1 \quad \text{and} \quad \frac{225}{256} \leq T_4(1) \leq 1. \quad (7)$$

These inequalities are sharp.

Proof. If $f \in \mathcal{S}_{\text{sech}}^*$, then $\zeta f'(\zeta)/f(\zeta) = \text{sech}(\omega(\zeta))$ for every $\zeta \in \mathbb{D}$, where ω is an analytic function. Since $p(\zeta) = (1 + \omega(\zeta))/(1 - \omega(\zeta)) \in \mathcal{P}$, we obtain

$$\frac{\zeta f'(\zeta)}{f(\zeta)} = \text{sech} \left[\frac{p(\zeta) - 1}{p(\zeta) + 1} \right]. \quad (8)$$

Applying some routine calculations, we get

$$\begin{aligned} & a_2\zeta + (2a_3 - a_2^2)\zeta^2 + (a_2^3 - 3a_2a_3 + 3a_4)\zeta^3 \\ & + (-a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2 + 4a_5)\zeta^4 + \dots \\ & = -\frac{1}{8}p_1^2\zeta^2 + \frac{1}{8}(p_1^3 - 2p_1p_2)\zeta^3 + \frac{1}{384}(-31p_1^4 + 144p_1^2p_2 - 96p_1p_3 - 48p_2^2)\zeta^4 + \dots \end{aligned}$$

To analyze the coefficients on both sides, we arrive the following:

$$a_2 = 0, \quad (9)$$

$$a_3 = -\frac{p_1^2}{16}, \quad (10)$$

$$a_4 = \frac{1}{24}p_1(p_1^2 - 2p_2), \quad (11)$$

$$a_5 = \frac{1}{384}(-7p_1^4 + 36p_1^2p_2 - 12p_2^2 - 24p_1p_3), \quad (12)$$

$$a_6 = \frac{1}{640}(3p_1^5 - 38p_1^3p_2 + 48p_1^2p_3 + 48p_1p_2^2 - 32p_1p_4 - 32p_2p_3). \quad (13)$$

Using (9) and (10), we obtain $T_3(1) = 1 - |p_1|^4/256$. Since the subclasses $\mathcal{S}_{\text{sech}}^*$ and $\mathcal{P}$ are rotationally invariant, we have $0 \leq p_1 \leq 2$, such that $p^2 =: x \in [0, 4]$. Thus, $T_3(1) = 1 - x^2/256$ for all $x \in [0, 4]$ and we obtain the extreme values of $T_3(1)$ as desired. The lower bound on $T_3(1)$ is sharp for the function $f_1(\zeta) = \zeta \exp\left(\int_0^\zeta \frac{\text{sech}(t)-1}{t} dt\right)$ or equivalently

$$f_1(\zeta) = \zeta - \frac{1}{4}\zeta^3 + \frac{13}{192}\zeta^5 \dots, \quad (14)$$

and the upper bound on $T_3(1)$ is sharp for the function $f_2(\zeta) = \zeta \exp\left(\int_0^\zeta \frac{\operatorname{sech}(t^2)-1}{t} dt\right)$ or equivalently

$$f_2(\zeta) = \zeta - \frac{1}{8}\zeta^5 + \dots \quad (15)$$

Using (9), (10) and (11) in the expression (2), we can obtain

$$T_4(1) = 1 + \frac{|p_1|^8}{16^4} - \frac{|p_1|^4}{128} - \frac{1}{24^2}|p_1|^2|p_1^2 - 2p_2|^2. \quad (16)$$

In view of Lemma 1, we have

$$|p_1^2 - 2p_2|^2 = p_1^4 + |2p_2|^2 - 2p_1^2 \operatorname{Re}(2\bar{p}_2) = (4 - p_1^2)^2 |\xi|^2 \quad (17)$$

for some $\xi \in \mathbb{D}$. From expressions (16) and (17), we can obtain

$$T_4(1) = 1 + \frac{1}{64} \left[\frac{1}{1024} p_1^8 - \frac{1}{2} p_1^4 - \frac{1}{9} p_1^2 (4 - p_1^2)^2 |\xi|^2 \right]. \quad (18)$$

Using the concept of rotationally invariant, we consider $p^2 =: x \in [0, 4]$ and $|\xi| =: y \in [0, 1]$. Therefore, we have

$$T_4(1) = 1 + \frac{1}{64} \left[\frac{x^4}{1024} - \frac{x^2}{2} - \frac{1}{9} x (4 - x)^2 y^2 \right].$$

A simple calculation obtains $\max T_4(1) = 1$ and $\min T_4(1) = 225/256$ in the region $[0, 4] \times [0, 1]$. The lower bound on $T_4(1)$ is most possible for the function f_1 given by (14), and the upper bound on $T_4(1)$ is most possible for the function f_2 given by (15). $\square$

If a function f is in $\mathcal{S}$, then $f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + \dots$. Therefore, the initial inverse coefficients are $A_2 = -a_2$, $A_3 = -a_3 + 2a_2^2$, $A_4 = -a_4 + 5a_2a_3 - 5a_2^3$ and $A_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4$ (see [32]). In terms of inverse coefficients, we have

$$H_2(3)(f^{-1}) = A_3A_5 - A_4^2 = a_3a_5 - a_4^2 - 3a_3^3.$$

In the next theorem, we obtain the bounds of $H_2(3)$ and $H_2(3)(f^{-1})$ for the functions $f \in \mathcal{S}_{\operatorname{sech}}^*$.

Theorem 2. Let $f \in \mathcal{S}_{\operatorname{sech}}^*$. Then,

$$|H_2(3)| \leq 0.021 \quad \text{and} \quad |H_2(3)(f^{-1})| \leq 0.02604.$$

Proof. If $f \in \mathcal{S}_{\operatorname{sech}}^*$, then, for every $\zeta \in \mathbb{D}$ and the Schwarz function ω , we have

$$\frac{\zeta f'(\zeta)}{f(\zeta)} = \operatorname{sech}(\omega(\zeta)). \quad (19)$$

Upon comparing the similar powers of ζ in the series expansion of the expression (19), we obtain

$$a_2 = 0, \quad (20)$$

$$a_3 = -\frac{c_1^2}{4}, \quad (21)$$

$$a_4 = -\frac{1}{3}c_1c_2, \quad (22)$$

$$a_5 = \frac{1}{24}(2c_1^4 - 3c_2^2 - 6c_1c_3). \quad (23)$$

In view of Equations (21)–(23), we can obtain

$$\begin{aligned} |H_2(3)| &= \frac{1}{864} |(-18c_1^6 - 69c_1^2c_2^2 + 54c_1^3c_3)| \\ &\leq \frac{1}{864} (18|c_1|^6 + 69|c_1|^2|c_2|^2 + 54|c_1|^3|c_3|). \end{aligned} \quad (24)$$

By using Lemma 4, we have

$$|H_2(3)| \leq \frac{1}{864} \left(18|c_1|^6 + 69|c_1|^2|c_2|^2 + 54|c_1|^3 \left(1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|} \right) \right).$$

Upon setting $x = |c_1|$ and $y = |c_2|$, the above expression becomes $|H_{2,3}(f)| \leq \Phi(x, y)$, where

$$\Phi(x, y) = \frac{1}{864} \left(18x^6 + 69x^2y^2 + 54x^3 \left(1 - x^2 - \frac{y^2}{1 + x} \right) \right).$$

In view of Lemma 4, we find the maximum value of the function Φ in the region $\Lambda = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2\}$. We consider two cases, as presented below:

(1) On the boundary of Λ , we have

$$\begin{aligned} \Phi(0, y) &= 0, \quad 0 \leq y \leq 1, \\ \Phi(x, 0) &= \frac{1}{48} (x^6 - 3x^5 + 3x^3) \leq \frac{1}{48}, \quad 0 \leq x \leq 1, \\ \Phi(x, 1 - x^2) &= \frac{1}{864} (33x^6 - 84x^4 + 69x^2) \leq 0.021, \quad 0 \leq x \leq 1. \end{aligned}$$

(2) In the interior of Λ , we have $\frac{\partial \Phi}{\partial y} = \frac{x^2y}{864} \left(138 - \frac{108x}{1+x} \right) \neq 0$. Therefore, the function Φ has no critical point in the interior of Λ .

Thus, in view of case (1) and case (2), we obtain the desired bound for $H_2(3)$.

Using Equations (21)–(23), we can obtain

$$\begin{aligned} |H_2(3)(f^{-1})| &= \frac{1}{1728} |(45c_1^6 - 138c_1^2c_2^2 + 108c_1^3c_3)| \\ &\leq \frac{1}{1728} (45|c_1|^6 + 138|c_1|^2|c_2|^2 + 108|c_1|^3|c_3|). \end{aligned} \quad (25)$$

Upon applying Lemma 4, we have

$$|H_2(3)(f^{-1})| \leq \frac{1}{1728} \left(45|c_1|^6 + 138|c_1|^2|c_2|^2 + 108|c_1|^3 \left(1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|} \right) \right).$$

Upon using $x = |c_1|$ and $y = |c_2|$, the above expression becomes $|H_{2,3}(f^{-1})| \leq \Psi(x, y)$, where

$$\Psi(x, y) = \frac{1}{1728} \left(45x^6 + 138x^2y^2 + 108x^3 \left(1 - x^2 - \frac{y^2}{1 + x} \right) \right).$$

Next, we determine the maximum value of the function Ψ in the region Λ . We consider two cases for Ψ :

(1) For the boundary of Λ , we obtain

$$\begin{aligned} \Psi(0, y) &= 0, \\ \Psi(x, 0) &= \frac{1}{1728} (45x^6 - 108x^5 + 108x^3) \leq \frac{5}{192}, \quad 0 \leq x \leq 1, \\ \Psi(x, 1 - x^2) &= \frac{1}{1728} (75x^6 - 168x^4 + 138x^2) \leq \frac{5}{192}, \quad 0 \leq x \leq 1. \end{aligned}$$

- (2) In the interior of Λ , we have $\frac{\partial \Psi}{\partial y} = \frac{x^2 y}{1728} \left(276 - \frac{216x}{1+x} \right) \neq 0$. Therefore, the function Ψ has no maximum value in the interior of Λ .

Thus, in view of case (1) and case (2), we obtain the desired bound for $H_2(3)(f^{-1})$. $\square$

Corollary 1. Let $f \in \mathcal{S}_{\text{sech}}^*$. Then,

$$|H_2(3)(f^{-1}) - H_2(3)| \leq \frac{3}{64}.$$

The inequality is sharp.

Proof. Since $H_2(3)(f^{-1}) = a_3 a_5 - a_4^2 - 3a_3^3 = H_2(3) - a_3^3$, we have

$$|H_2(3)(f^{-1}) - H_2(3)| = 3|a_3^3|.$$

From (21), we get

$$|H_2(3)(f^{-1}) - H_2(3)| \leq \frac{3}{64} |c_1|^6 \leq \frac{3}{64}, \quad 0 \leq |c_1| \leq 1.$$

Equality holds for the extremal function given by $f_h(\zeta) = \zeta - \frac{1}{4}\zeta^3 + \frac{13}{192}\zeta^5 + \dots$. $\square$

Next, we obtain the bounds of the Hankel determinants $H_3(1)$ and $H_3(2)$ for the functions $f \in \mathcal{S}_{\text{sech}}^*$.

Theorem 3. Let $f \in \mathcal{A}$ be in the class $\mathcal{S}_{\text{sech}}^*$. Then,

$$|H_3(1)| \leq \frac{23}{288} + \frac{3}{4\sqrt{137}} \approx 0.143938 \quad \text{and} \quad |H_3(2)| \leq \frac{527}{8640} \approx 0.0609954.$$

Proof. Since $f \in \mathcal{S}_{\text{sech}}^*$; then, $a_2 = 0$, so that $H_3(1) = -a_3^3 - a_4^2 + a_3 a_5$ and $H_3(2) = a_4(a_3 a_5 - a_4^2) - a_3(a_3 a_6 - a_4 a_5)$. Upon setting the values of a_i 's ($i = 3, 4, 5$) given by (10)–(12) into the expression $H_3(1)$, we obtain

$$\begin{aligned} H_3(1) &= \frac{1}{36,864} (-136p_1^6 + 40p_1^4 p_2 - 184p_1^2 p_2^2 + 144p_1^3 p_3) \\ &= \frac{1}{36,864} (13p_1^3 \alpha_1 + 184p_1^2 p_2 \alpha_2), \end{aligned} \quad (26)$$

where $\alpha_1 = (144/13)p_3 - p_1^3$ and $\alpha_2 = (5/23)p_1^2 - p_2$. Using Lemma 2 and Lemma 3, we can obtain $|\alpha_1| = |(144/13)p_3 - p_1^3| \leq 3456/7\sqrt{137}$ and $|\alpha_2| = |(5/23)p_1^2 - p_2| \leq 2$. Using triangle inequality in the expression (26) and in view of the above inequalities, we can obtain the desired estimate of $H_3(1)$.

Further, upon setting the values of a_i 's ($i = 3, 4, 5, 6$) given by (10)–(13) into the expression $H_3(2)$, we obtain

$$\begin{aligned} H_3(2) &= \frac{1}{4,423,680} (p_1^3 (19p_1^6 - 54p_1^4 p_2 + 144p_1^3 p_3 - 96p_1^2 (p_2^2 - 9p_4) - 2016p_1 p_2 p_3 + 1120p_2^3)) \\ &= \frac{1}{4,423,680} (54p_1^7 \alpha_3 + 1120p_1^3 p_2^2 \alpha_4 + 2016p_1^4 p_3 \alpha_5 + 864p_1^5 p_4), \end{aligned}$$

so that

$$|H_3(2)| \leq \frac{1}{4,423,680} (54|p_1|^7 |\alpha_3| + 1120|p_1|^3 |p_2|^2 |\alpha_4| + 2016|p_1|^4 |p_3| |\alpha_5| + 864|p_1|^5 |p_4|), \quad (27)$$

where $\alpha_3 = (19/54)p_1^2 - p_2$, $\alpha_4 = (-3/35)p_1^2 + p_2$ and $\alpha_5 = (1/14)p_1^2 - p_2$. Using Lemma 2, we can obtain $|\alpha_3| \leq 2$, $|\alpha_4| \leq 2$ and $|\alpha_5| \leq 2$. Therefore, by using $|p_n| \leq 2$ for all $n \geq 1$, the inequality (27) yields the desired bound for $H_3(2)$ $\square$.

We now provide a bound of the Zalcman conjecture $|a_5 - a_3^2|$ for the functions $f \in \mathcal{S}_{\text{sech}}^*$.

Theorem 4. Let $f \in \mathcal{S}_{\text{sech}}^*$. Then, $|a_5 - a_3^2| \leq \frac{47}{48}$.

Proof. Since $f \in \mathcal{S}_{\text{sech}}^*$, in view of (10) and (12), we have

$$a_5 - a_3^2 = \frac{1}{768} (-17p_1^4 + 72p_1^2p_2 - 48p_1p_3 - 24p_2^2)$$

so that

$$|a_5 - a_3^2| \leq \frac{1}{768} (-17|p_1|^4 + 48|p_1||\alpha_3| + 24|p_2|^2), \quad (28)$$

where $\alpha_3 = \frac{3}{2}p_1p_2 - p_3$. Using Lemma 2, we have $|\alpha_3| \leq 4$. Thus, using inequalities $|p_n| \leq 2$ for all $n \geq 1$ and $|\alpha_3| \leq 4$ in (28), we obtain the desired estimate of $|a_5 - a_3^2|$. $\square$

The Bernardi integral operator $\mathcal{L}_\sigma : \mathcal{A} \rightarrow \mathcal{A}$ ($\sigma > 0$) is defined by

$$\mathcal{L}_\sigma f(\zeta) = \frac{1+\sigma}{\zeta^\sigma} \int_0^\zeta t^{\sigma-1} f(t) dt.$$

From this operator, the following recurrence formula can easily be obtained:

$$\zeta(\mathcal{L}_\sigma f(\zeta))' = (1+\sigma)f(\zeta) - \sigma\mathcal{L}_\sigma f(\zeta). \quad (29)$$

Using a technique introduced by Miller and Mocanu [13,33], we establish some subordination properties. Thus, we need the following lemmas:

Lemma 5 ([34]). Let κ ($\kappa(0) = 1$) be a convex univalent in $\mathbb{D}$, and let ϕ of the form $\phi(\zeta) = 1 + c_1\zeta \cdots$ be analytic in $\mathbb{D}$. If

$$\phi(\zeta) + \frac{1}{\mu} \zeta \phi'(\zeta) \prec \kappa(\zeta), \quad (\text{Re}(\mu) \geq 0; \mu \neq 0)$$

then

$$\phi(\zeta) \prec \tau(\zeta) = \frac{\mu}{\zeta^\mu} \int_0^\zeta t^{\mu-1} \kappa(t) dt \prec \kappa(\zeta), \quad (30)$$

and τ is the best dominant of (30).

Lemma 6 ([33]). Let γ ($\gamma \neq 0$) and μ be complex numbers, and let κ ($\kappa(0) = 1$) be a convex univalent in $\mathbb{D}$ satisfying $\text{Re}(\gamma\kappa(\zeta) + \mu) > 0$. Let ϕ be the analytic in $\mathbb{D}$ and satisfy the subordination given by (6). If the Briot–Bouquet differential equation given by

$$q(\zeta) + \frac{\zeta q'(\zeta)}{\gamma q(\zeta) + \mu} = \kappa(\zeta), \quad (q(0) = 1) \quad (31)$$

has a univalent solution q ; then, $\phi(\zeta) \prec q(\zeta) \prec \kappa(\zeta)$, and q is the best dominant of (6). The differential Equation (31) has a formal solution, given by

$$q(\zeta) = z^\mu [\Theta(\zeta)]^\gamma \left(\gamma \int_0^\zeta [\Theta(t)]^\gamma t^{\mu-1} dt \right)^{-1} - \mu/\gamma, \quad (32)$$

where

$$\Theta(\zeta) = \zeta \exp \int_0^\zeta \frac{\kappa(t) - 1}{t} dt.$$

In Theorem 5, we provide a Briot–Bouquet differential subordination relation by using the Bernardi integral operator. In Theorem 6, we find a univalent solution to the Briot–Bouquet differential equation, and we observe that this solution is the best possible solution to the Briot–Bouquet differential subordination for the class $\mathcal{S}_{\text{sech}}^*$.

Theorem 5. Let $0 < \delta < 1$, and $\vartheta \geq 1$. If $f \in \mathcal{A}$ holds,

$$(1 - \delta) \frac{f(\zeta)}{\zeta} + \delta \frac{\mathcal{L}_\sigma f(\zeta)}{\zeta} \prec \text{sech}(\zeta), \quad (33)$$

then

$$\text{Re} \left\{ \left(\frac{\mathcal{L}_\sigma f(\zeta)}{\zeta} \right)^{1/\vartheta} \right\} > \left(\frac{1 + \sigma}{1 - \delta} \int_0^1 u^{\frac{1+\sigma}{1-\delta}-1} \text{sech}(u) du \right)^{1/\vartheta}. \quad (34)$$

The result is sharp.

Proof. Let $\phi(\zeta) = \frac{\mathcal{L}_\sigma f(\zeta)}{\zeta}$, ($\zeta \in \mathbb{D}$) with $\phi(0) = 1$. Using (29), we can obtain

$$\frac{f(\zeta)}{\zeta} = \phi(\zeta) + \frac{1}{1 + \sigma} \zeta \phi'(\zeta).$$

Applying (33), we conclude that

$$(1 - \delta) \frac{f(\zeta)}{\zeta} + \delta \frac{\mathcal{L}_\sigma f(\zeta)}{\zeta} = \phi(\zeta) + \frac{1 - \delta}{1 + \sigma} \zeta \phi'(\zeta) \prec \text{sech}(\zeta).$$

From Lemma 5, we have

$$\phi(\zeta) \prec \frac{1 + \sigma}{1 - \delta} \zeta^{-\frac{1+\sigma}{1-\delta}} \int_0^\zeta t^{\frac{1+\sigma}{1-\delta}-1} \text{sech}(t) dt$$

or

$$\frac{\mathcal{L}_\sigma f(\zeta)}{\zeta} = \frac{1 + \sigma}{1 - \delta} \int_0^1 u^{\frac{1+\sigma}{1-\delta}-1} \text{sech}(u\omega(\zeta)) du,$$

where ω is an analytic function. Since $\text{Re}(\text{sech}(\omega(\zeta))) > \text{sech}(r)$, we write

$$\text{Re}(\text{sech}(u\omega(\zeta))) > \text{sech}(ur).$$

Letting $r \rightarrow 1^-$, we can obtain

$$\text{Re} \left(\frac{\mathcal{L}_\sigma f(\zeta)}{\zeta} \right) > \frac{1 + \sigma}{1 - \delta} \int_0^1 u^{\frac{1+\sigma}{1-\delta}-1} \text{sech}(u) du > 0, \quad (\zeta \in \mathbb{D}). \quad (35)$$

Since $\text{Re}(\omega^{1/\vartheta}) \geq \text{Re}(\omega)^{1/\vartheta}$ for $\text{Re}(\omega) > 0$ and $\vartheta \geq 1$, using (35), we prove (34). Sharpness follows for the function

$$\frac{\mathcal{L}_\sigma f(\zeta)}{\zeta} = \frac{1 + \sigma}{1 - \delta} \int_0^1 u^{\frac{1+\sigma}{1-\delta}-1} \text{sech}(u\zeta) du$$

so that

$$(1 - \delta) \frac{f(\zeta)}{\zeta} + \delta \frac{\mathcal{L}_\sigma f(\zeta)}{\zeta} = \text{sech}(\zeta),$$

and

$$\frac{\mathcal{L}_\sigma f(\zeta)}{\zeta} \rightarrow \frac{1+\sigma}{1-\delta} \int_0^1 u^{\frac{1+\sigma}{1-\delta}-1} \operatorname{sech}(u\zeta) du$$

as $\zeta \rightarrow 1^-$. $\square$

Theorem 6. Let $\mathcal{L}_\sigma f(\zeta) \neq 0$, and $\sigma > 0$. If $f \in \mathcal{S}_{\operatorname{sech}}^*$ and $\operatorname{Re}(\operatorname{sech}(\zeta) + \sigma) > 0$, then $\mathcal{L}_\sigma f \in \mathcal{S}_{\operatorname{sech}}^*$. Furthermore, if $f \in \mathcal{S}_{\operatorname{sech}}^*$, then

$$\frac{\zeta(\mathcal{L}_\sigma f(\zeta))'}{\mathcal{L}_\sigma f(\zeta)} \prec q(\zeta) \prec \operatorname{sech}(\zeta), \quad (36)$$

where

$$q(\zeta) = \zeta^{\sigma+1} e^{K(\zeta)} \left(\int_0^\zeta t^\sigma e^{K(t)} dt \right)^{-1} - \sigma, \quad (37)$$

where $K(\zeta) = \int_0^\zeta \frac{\operatorname{sech}(t)-1}{t} dt$ and $K(t) = \int_0^t \frac{\operatorname{sech}(s)-1}{s} ds$, and q is the best dominant of (36).

Proof. Let

$$\phi(\zeta) = \frac{\zeta(\mathcal{L}_\sigma f(\zeta))'}{\mathcal{L}_\sigma f(\zeta)}, \quad (\zeta \in \mathbb{D})$$

with $\phi(0) = 1$. Using (29), we get

$$(1+\sigma) \frac{f(\zeta)}{\mathcal{L}_\sigma f(\zeta)} = \phi(\zeta) + \sigma.$$

Upon differentiating logarithmically, we obtain

$$\frac{\zeta f'(\zeta)}{f(\zeta)} = \phi(\zeta) + \frac{\zeta \phi'(\zeta)}{\phi(\zeta) + \sigma} \prec \operatorname{sech}(\zeta). \quad (38)$$

Consider the equation

$$q(\zeta) + \frac{\zeta q'(\zeta)}{q(\zeta) + \sigma} = \phi(\zeta) := \operatorname{sech}(\zeta), \quad (39)$$

where the function q is analytic, satisfying $q(0) = 1$, and the function $\kappa(\zeta) = \operatorname{sech}(\zeta)$ is a convex univalent in $\mathbb{D}$. Let $P(\zeta) = \gamma \kappa(\zeta) + \mu$. According to (39) and Lemma 6, we can observe that $\gamma = 1, \mu = \sigma$ and $P(\zeta) = \operatorname{sech}(\zeta) + \sigma$. For proving $\operatorname{Re}(\operatorname{sech}(\zeta) + \sigma) > 0$, it is enough to set $\zeta = e^{ix}$, $x \in [0, \pi]$ into $\operatorname{sech}(\zeta)$, and we can obtain

$$\operatorname{sech}(e^{ix}) = \frac{\cosh(\cos x) \cos(\sin x) - i \sinh(\cos x) \sin(\sin x)}{\sinh^2(\cos x) + \cos^2(\sin x)}$$

and

$$\operatorname{Re}(\operatorname{sech}(\zeta) + \sigma) = \frac{\cosh(\cos x) \cos(\sin x)}{\sinh^2(\cos x) + \cos^2(\sin x)} + \sigma > 0$$

under the condition $\sigma > 0$. Hence, there is a univalent solution of the Equation (39). To get this solution, we apply the Lemma 6. Since $\kappa(\zeta) = \operatorname{sech}(\zeta)$, we find

$$\Theta(\zeta) = \zeta \exp \int_0^\zeta \frac{\kappa(t)-1}{t} dt = \zeta \exp \int_0^\zeta \frac{\operatorname{sech}(t)-1}{t} dt.$$

Combining the above result, together with $\gamma = 1$ and $\mu = \sigma$, into Formula (32), we obtain (37). This is the univalent solution of (39). Since ϕ is an analytic function satisfying the relation (38), then $\phi(\zeta) \prec q(\zeta) \prec \kappa(\zeta) := \operatorname{sech}(\zeta)$. $\square$

Finally, we find the majorization property for functions $f \in \mathcal{S}_{\text{sech}}^*$. Hence, we need the following lemma.

Lemma 7 ([35]). *If the function ϕ is bounded and analytic in $\mathbb{D}$, then*

$$|\phi'(\zeta)| \leq \frac{1 - |\phi(\zeta)|^2}{1 - |\zeta|^2},$$

where $|\zeta| < 1$.

Theorem 7. *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_{\text{sech}}^*$. If $f(\zeta) \ll g(\zeta)$ for every $\zeta \in \mathbb{D}$, then $|f'(\zeta)| \leq |g'(\zeta)|$ for $|\zeta| \leq r_1$, where r_1 is the smallest positive root of the equation*

$$(1 - r^2)\text{sech}(r) - 2r = 0.$$

Proof. Since f is majorized by g , there exists a function ϕ in $\mathbb{D}$ satisfying $|\phi(\zeta)| \leq 1$ so that

$$f(\zeta) = \phi(\zeta)g(\zeta). \quad (40)$$

On differentiating both sides of (40), we obtain

$$f'(\zeta) = \phi'(\zeta)g(\zeta) + \phi(\zeta)g'(\zeta). \quad (41)$$

Since $g \in \mathcal{S}_{\text{sech}}^*$, then

$$\frac{\zeta g'(\zeta)}{g(\zeta)} = \text{sech}(\omega(\zeta)), \quad (42)$$

where ω is a function satisfying $\omega(0) = 0$ and $|\omega(\zeta)| \leq |\zeta|$ in $\mathbb{D}$. Let $\omega(\zeta) = Re^{ix}$ so that $R \leq |\zeta| = r$, and $-\pi \leq x \leq \pi$. Hence,

$$|\text{sech}(Re^{ix})|^2 = \left| \frac{\cosh(R \cos x) \cos(R \sin x) - i \sinh(R \cos x) \sin(R \sin x)}{\sinh^2(R \cos x) + \cos^2(R \sin x)} \right|^2 =: \Omega(x).$$

However, the equation $\Omega'(x) = 0$ has roots $0, \pm\pi$ and $\pm\pi/2$ in the interval $[-\pi, \pi]$; it is sufficient to consider the roots in $[0, \pi]$ because $\Omega(x)$ is symmetric with respect to the real axis. Then, we have

$$\min\{\Omega(0), \Omega(\pi)\} = \text{sech}^2(R), \quad \max\{\Omega(\pi/2)\} = \sec^2(R)$$

so that

$$\text{sech}(r) \leq \text{sech}(R) \leq |\text{sech}(\omega(\zeta))| \leq \text{sech}(R) \leq \text{sech}(r).$$

By using (41), (42) and the above relation together with Lemma 7, we obtain

$$|f'(\zeta)| \leq \left[\frac{(1 - |\phi(\zeta)|^2)}{(1 - r^2)} \frac{r}{\text{sech}(r)} + |\phi(\zeta)| \right] |g'(\zeta)|, \quad (|\zeta| = r < 1). \quad (43)$$

On setting $|\phi(\zeta)| = \rho$ ($0 \leq \rho \leq 1$) into (43), we obtain $|f'(\zeta)| \leq Y(r, \rho)|g'(\zeta)|$, where

$$Y(r, \rho) = \frac{(1 - \rho^2)}{(1 - r^2)} \frac{r}{\text{sech}(r)} + \rho.$$

In order to obtain r_1 , we take

$$r_1 = \max\{r \in (0, 1) : Y(r, \rho) \leq 1\} = \max\{r \in (0, 1) : \psi(r, \rho) \geq 0\}$$

for all $\rho \in [0, 1]$, where $\psi(r, \rho) = (1 - r^2)\operatorname{sech}(r) - r(1 + \rho)$. Since $\frac{\partial}{\partial \rho}\psi(r, \rho) = -r < 0$, then $\min\{\psi(r, \rho) \geq 0, \rho \in [0, 1]\} = \psi(r, 1) := \psi(r)$, where $\psi(r) = (1 - r^2)\operatorname{sech}(r) - 2r$. Since $\psi(0) = 1 > 0$, and $\psi(1) = -2 < 0$, there exists a smallest positive root r_1 , such that $\psi(r) \geq 0$ for all $r \in [0, r_1]$. $\square$

3. Conclusions

In this article, we determined sharp estimates of the Hermitian–Toeplitz determinants $T_3(1)$ and $T_4(1)$ for the class of starlike functions defined by the hyperbolic secant function. We computed a bound of the Hankel determinant $H_2(3)$ involving initial coefficients and initial inverse coefficients. We also computed the bounds of the Hankel determinants $H_3(1)$ and $H_3(2)$, and a bound of the generalized Zalcman conjecture. Moreover, we established a Briot–Bouquet differential subordination relation by using the Bernardi integral operator as well as obtaining a univalent solution to the Briot–Bouquet differential equation. Finally, we investigated the majorization properties.

The idea used in this paper can be extended to resolve some other problems. For instance, an estimate of fourth-order Hankel determinants can be obtained for such functions. Furthermore, these types of results can be investigated for other subclasses whose image domain lies in other different trigonometric functions.

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