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REVERSE PERIOD DOUBLINGS IN A 3^{RD} ORDER NONLINEAR AND AUTONOMOUS ELECTRIC CIRCUIT

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Abstract

We have studied the dynamics of Chua's canonical circuit, when the v-i characteristic of the nonlinear resistor of the circuit is a smooth cubic function. Unlike the monotone bifurcation behavior of the members of Chua's circuit family with a piecewise linear resistor, reverse period doublings, as a parameter of the circuit is varied in a monotone way, have been observed in the circuit we have studied. Dynamics of the circuit is very sensitive to initial conditions, as chaotic attractors coexist with period-1 limit cycles.

Key-words: *Nonlinear electric circuit, chaos, antimonotonicity, cubic nonlinearity, bubbles, crisis, coexisting attractors.*

1. Introduction

Electric circuits have emerged as a simple yet powerful experimental and analytical tool in studying chaotic behavior in nonlinear dynamics. Most chaotic and bifurcation effects cited in the literature have been observed in electric circuits e.g. the period-doubling route to chaos [1-4], intermittency route to chaos [5-8], quasiperiodicity route to chaos [9-11], crisis [12-14], antimonotonicity [15, 16]. Chua's circuit is a paradigm for chaos [17]. Among the members of Chua's circuit family, the autonomous canonical Chua's circuit introduced by Chua and Lin [18] is of considerable importance. This is because it is capable of realizing the behavior of every member of the Chua's circuit family [18, 8, 14]. It consists of two active elements, one linear negative conductor, and one nonlinear resistor with odd-symmetric piecewise linear υ-i characteristic.

Later, some bifurcation phenomena were obtained for Chua's circuit with a smooth nonlinearity, in particular, cubic nonlinearity [19, 20]. The choice of a cubic nonlinearity has several advantages over a piecewise-linear one. It does not require absolute-valued functions and it is smooth, which is desirable from a mathematical perspective. Moreover, all phenomena found in the piecewise linear version also exist in the cubic version.

2. The Canonical Chua's Circuit

Chua's canonical circuit is a nonlinear autonomous 3^{rd}-order electric circuit (Figure 1a). The nonlinear element is a nonlinear resistor, while G_n is a linear negative conductance. In this paper, the v-i characteristic of the nonlinear resistor is a smooth cubic function, Figure 1b, of the form

$$i_N = -k_1 v_{C1} + k_3 v_{C1}^3 \quad (1)$$

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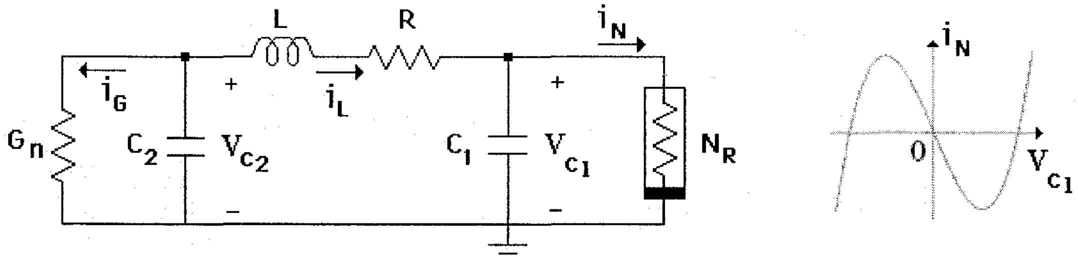
where $k_1, k_3 > 0$, instead of the piecewise linear type-N characteristic used in the previous studies [8, 14, 18]. The laboratory realization of this nonlinear resistor can be found in [19].



(a) (b)

Figure 1. (a) Chua's canonical circuit, (b) the cubic v-i characteristic of the nonlinear resistor.

The state equations of the circuit are the following:

$$\frac{dv_{C1}}{dt} = \frac{1}{C_1}\left(i_L + k_1 v_{C1} - k_3 v_{C1}^3\right) \quad (2)$$

$$\frac{dv_{C2}}{dt} = -\frac{1}{C_2}\left(i_L + G_n v_{C2}\right) \quad (3)$$

$$\frac{di_L}{dt} = \frac{1}{L}\left(-v_{C1} + v_{C2} - Ri_L\right) \quad (4)$$

We have chosen the following values for the circuit parameters: L = 100 mH, R = 330 Ω, and $G_n = 0.40$ mS, while $k_1 = 0.3$ mS and $k_3 = 0.1$ mA/V^3. Giving constant values to capacitance C_1, we have plotted the bifurcation diagrams v_{C1} vs. C_2. The comparative study of the bifurcation diagrams gives the qualitative changes of the dynamics of the system, as C_1 takes different discrete values.

3. The Period – Doubling Route to Chaos

Cascades of period-doubling bifurcations have long been recognized to be one of the most common routes to chaos, as exemplified e.g. by the one-dimensional (1D) logistic map $x_{n+1} = \lambda x_n(1 - x_n)$. As the parameter λ in such a map is increased, it is known that periodic orbits are only created but never destroyed, Figure 2.

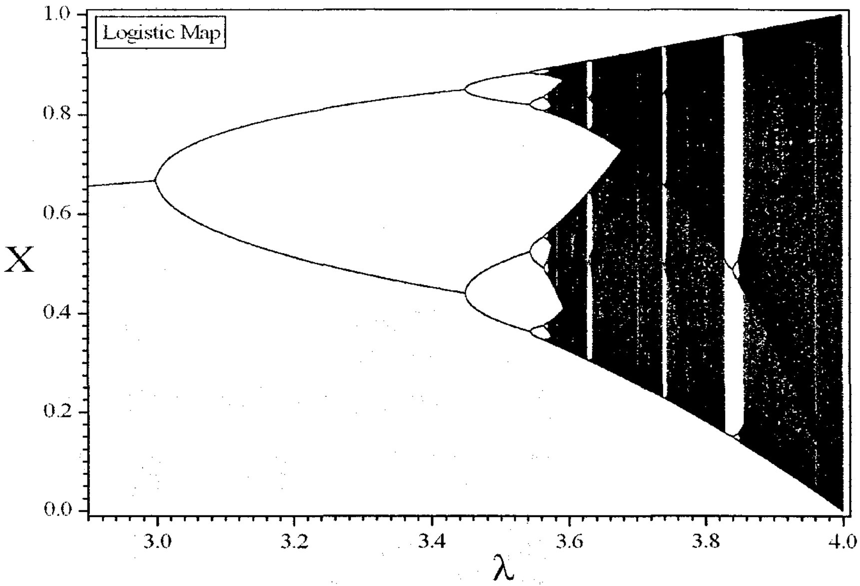


Figure 2. The bifurcation diagram of the logistic map.

4. Antimonotonicity

Unlike the monotone bifurcation behavior of the logistic map, however it has been shown, that in many common nonlinear dynamical systems, periodic orbits can be both created as well as destroyed, via reverse bifurcation sequences as a parameter is varied in a monotone way. Dawson et al., [21], named this type of creation and annihilation of periodic orbits *antimonotonicity*.

Reversals of period-doubling cascades have been observed in various nonlinear physical systems both numerically and experimentally. In one of the first studies of this phenomenon [22], the occurrence of such reverse sequences was connected to the dynamics of a cubic 1D map. As examples of numerical simulations, we cite the van der Pol equation [23], Duffing's oscillator [24], a RC-ladder chaos generator [25], and an autonomous 4th-order nonlinear electric circuit [26]. Experimental manifestations of antimonotonicity have also been observed on the driven R, L,p-n junction nonlinear circuit [2, 27, 28], and on Chua's circuit, with an asymmetric v-i characteristic [29].

The general form of the bifurcation diagram in the case of antimonotonicity, is shown in Figure 3. The system, starting from a period$\tilde{N}$ state, following the period doubling route enters the chaotic state. The system, as the control parameter "b" varies in a monotone way, leaves the chaotic regime via a reverse period doubling cascade ending in period$\tilde{N}$ state. This configuration is known as "period$\tilde{N}$ bubble".

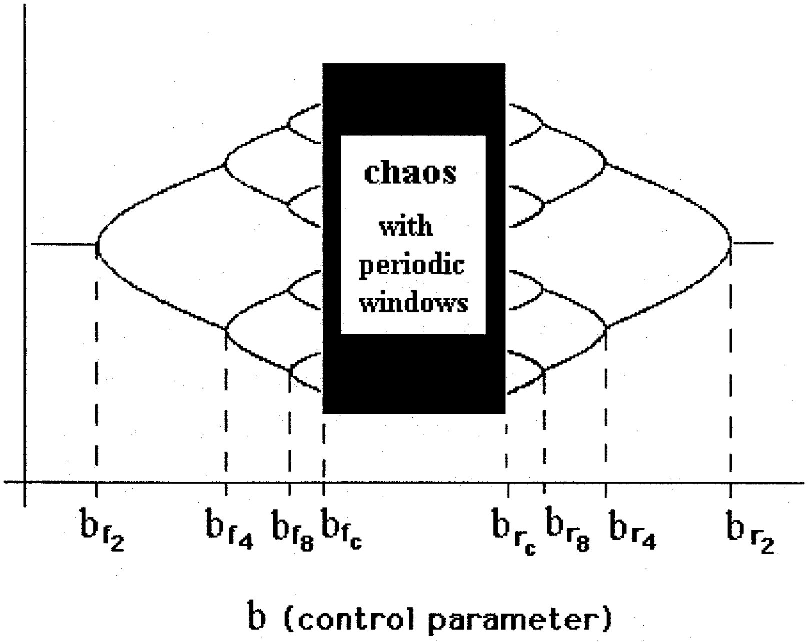


Figure 3. The general scheme of a period – 1 bubble.

In this paper, we have studied the dynamics of Chua's canonical circuit [18] with an odd symmetric cubic nonlinearity and we have focused on the phenomenon of antimonotonicity, which has never observed in the members of Chua's circuit family with a piecewise linear symmetric i-v characteristic.

5. Period – 1 Bubbles

The bifurcation diagram v_{C1} vs. C_2, for $C_1 = 41.0$ nF is shown in Figure 4. As C_2 is decreased, the system always remains in a periodic state following the scheme: period1 → period2 → period1 (or p1 → p2 → p1). Bier and Bountis, [22], named this scheme "primary bubble". The bifurcation diagram v_{C1} vs. C_2, for $C_1 = 40.0$ nF is shown in Figure 5. The system remains again in a periodic state, but a period4 state is now formed.

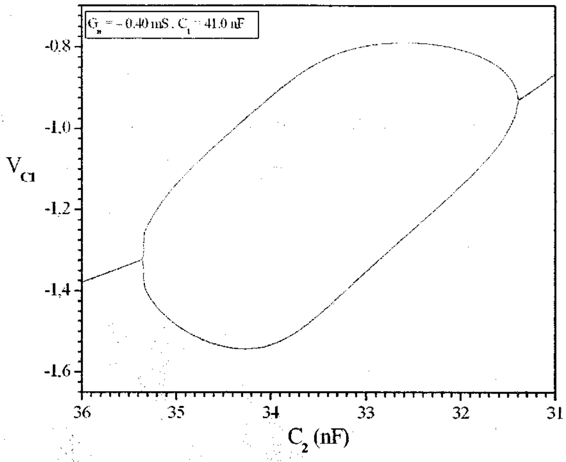


Figure 4. The bifurcation diagram v_{C1} vs. C_2, for $C_1 = 41.0$ nF.

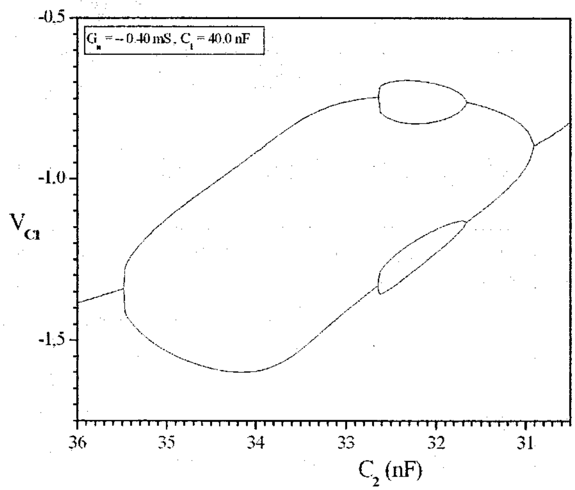


Figure 5. The bifurcation diagram v_{C1} vs. C_2, for $C_1 = 40.0$ nF.

As C_2 is decreased, chaotic states appear, as we can observe in Figure 6, where the bifurcation diagram v_{C1} vs. C_2, for C_1 = 39.5 nF is shown. The bubble is now *chaotic*. Chaotic states become enlarged, as C_2 is decreased (Figures 7-10). For all the bubbles, the initial and the final dynamic state is a period1 state, so the bubbles are “period1 bubbles”.

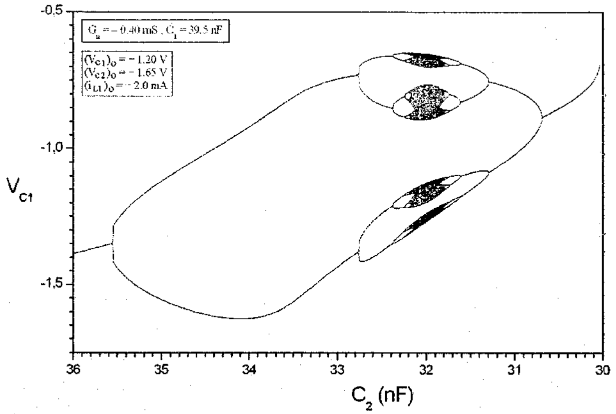


Figure 6. The bifurcation diagram v_{C1} vs. C_2, for $C_1 = 39.5$ nF.

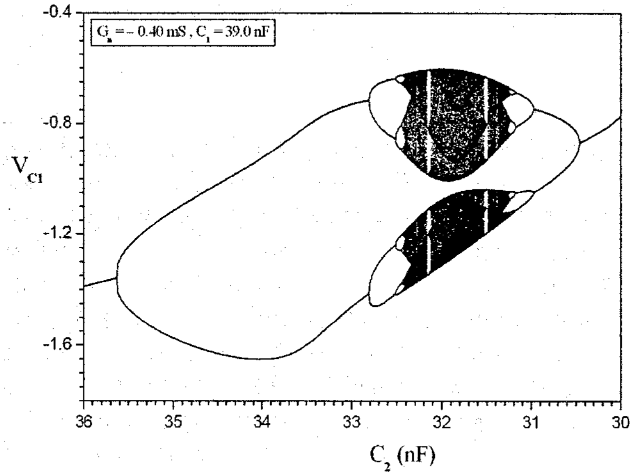


Figure 7. The bifurcation diagram v_{C1} vs. C_2, for $C_1 = 39.0$ nF.

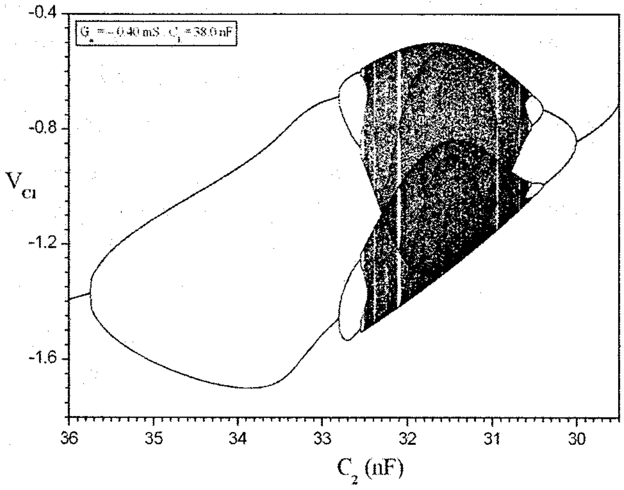


Figure 8. The bifurcation diagram v_{C1} vs. C_2, for $C_1 = 38.0$ nF.

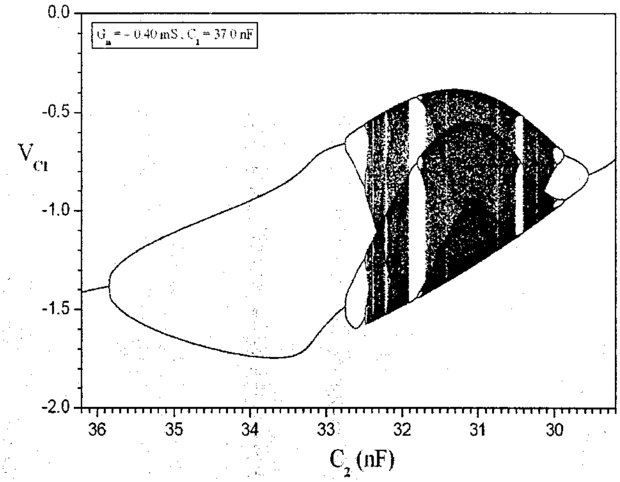


Figure 9. The bifurcation diagram v_{C1} vs. C_2, for $C_1 = 37.0$ nF.

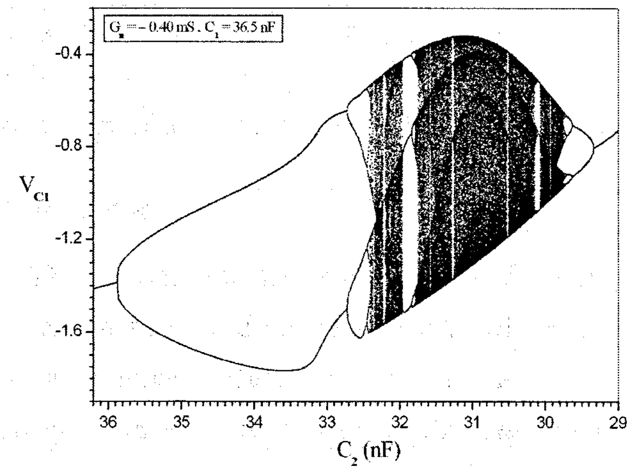


Figure 10. The bifurcation diagram v_{C1} vs. C_2, for $C_1 = 36.5$ nF.

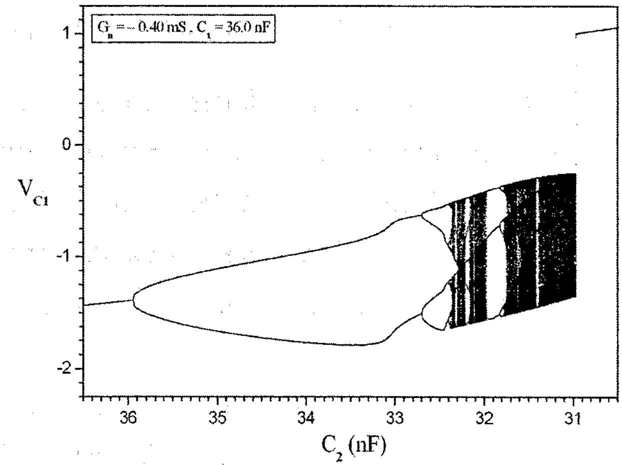


Figure 11. The bifurcation diagram v_{C1} vs. C_2, for $C_1 = 36.0$ nF.

Reverse period doublings are destroyed, when $C_1 = 36.0$ nF (Figure 11). A sudden transition, from a chaotic to a periodic state is observed at $C_2 = 31.0$ nF, a phenomenon called *crisis,*[30]. In Figure 12, the chaotic spiral attractor ($C_2 = 31.0$ nF) is shown, while in Figure 13 the periodic attractor ($C_2 = 30.8$ nF) is shown.

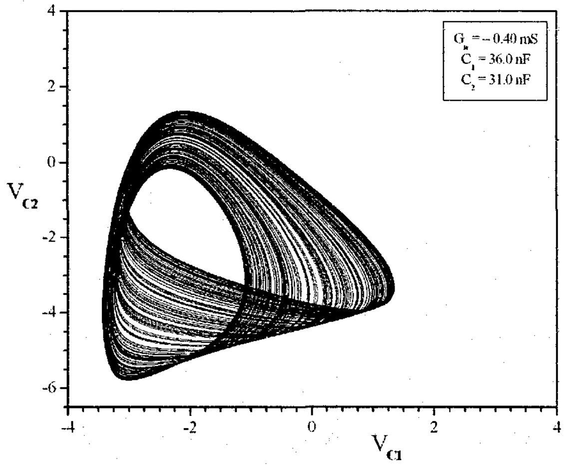


Figure 12. Phase portrait for $C_1 = 36.0$ nF and $C_2 = 31.0$ nF. Chaotic spiral attractor.

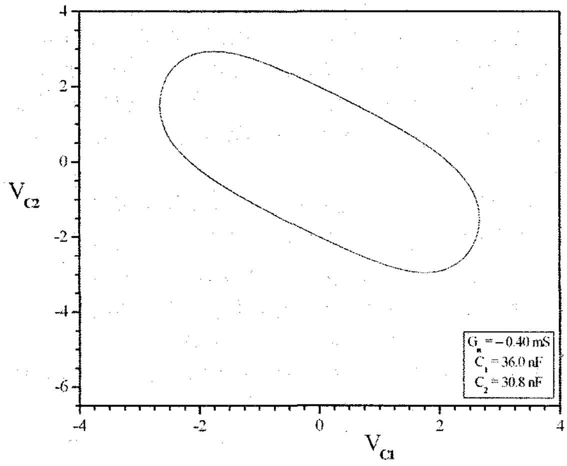


Figure 13. Phase portrait for $C_1 = 36.0$ nF and $C_2 = 30.8$ nF. Limit cycle.

6. Period – N Bubbles

For lower values of C_1, bubbles of higher period are formed. In Figure 14, we can observe a chaotic bubble of period–2, while in Figure 15, a chaotic bubble of period–3 is shown. For $G_n \overset{-}{=}$ 0.45 mS and $C_1 = 40.0$ nF , a period–5 chaotic bubble is formed, as we can see in Figure 16.

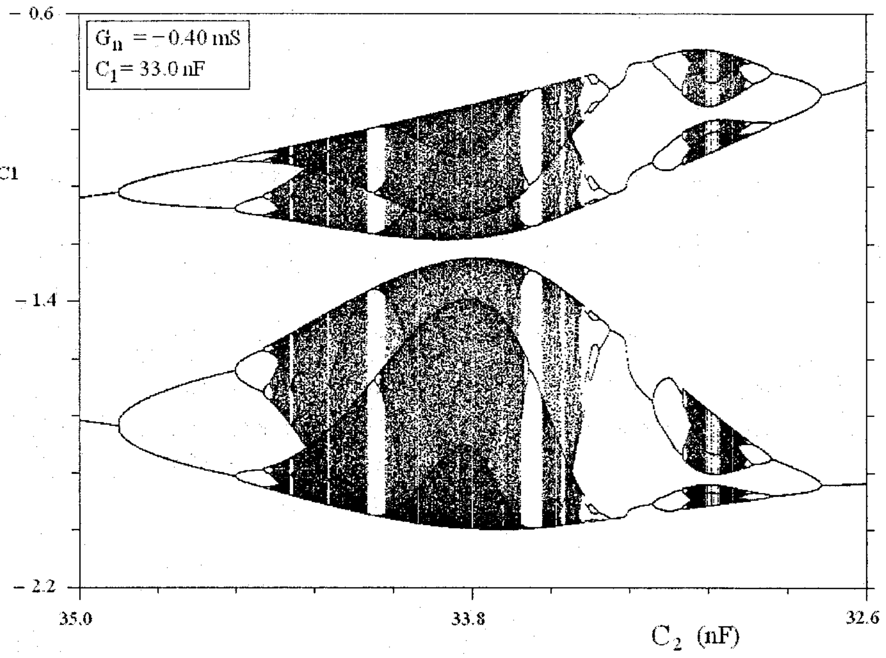


Figure 14. Chaotic bubble of period2 for $G_n = -$ 0.40 mS and $C_1 = 33.0$ nF.

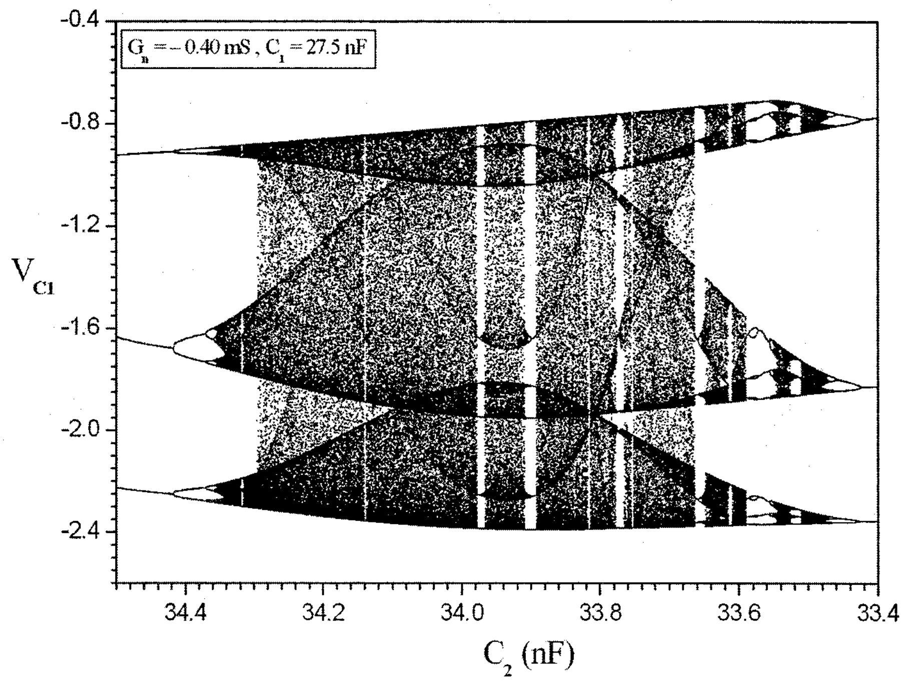


Figure 15. Chaotic bubble of period3 for $G_n = -$ 0.40 mS and $C_1 = 27.5$ nF.

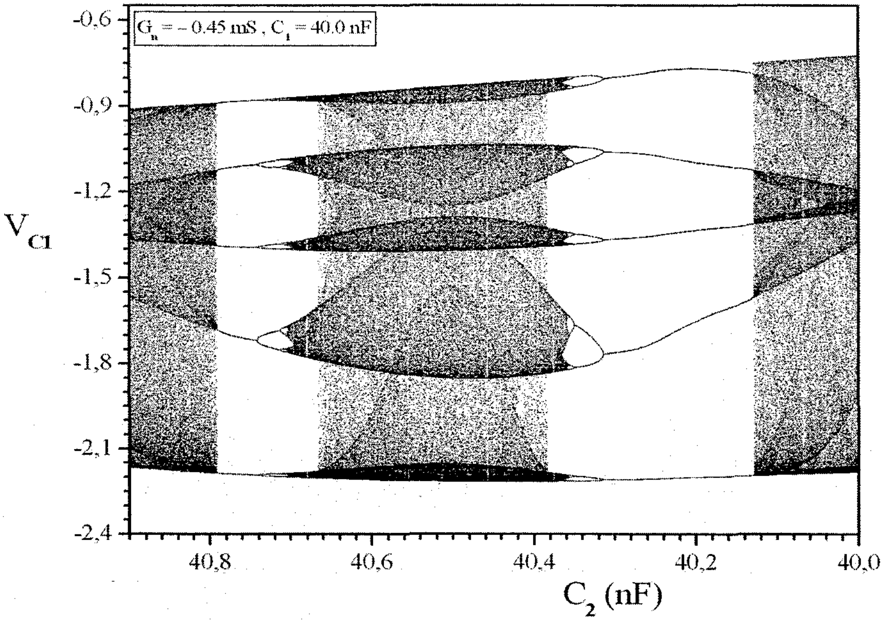


Figure 16. Chaotic bubble of period5 for $G_n = -$ 0.45 mS and $C_1 = 40.0$ nF.

7. Discussion and Conclusions

Bier and Bountis, [22], demonstrated that reverse period doubling sequences are expected to occur, when a minimum number of conditions is fulfilled. Their main result was, that a reverse period doubling sequence is likely to occur in any nonlinear system, where there is a symmetry transformation, under which the state equation remains invariant.

Indeed our system of differential equations (2-4) under the transformation

$$v_{C1} \rightarrow -v_{C1}\ ,\ v_{C2} \rightarrow -v_{C2}\ ,\ i_L \rightarrow -i_L \quad (5)$$

remains invariant. In addition, it has also been demonstrated in the literature, [22, 31], that reverse period doubling commonly arises in nonlinear dynamical systems involving the variation of two parameters. It is important, however, that the period doubling "trees" develop symmetrically *towards* each other along some line in parameter space. This would allow them to terminate, by joining their "branches" to form "bubbles", thus exhibiting the phenomenon of antimonotonicity.

We have to notice, that antimonotonicity is present, when the well-known "double-scroll" Chua's attractor, [32], is *absent*. This is an explanation, that antimonotonicity has not been observed in Chua's circuit with a piecewise linear resistor, although the former criteria are fulfilled.

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