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Structure Preserving Schemes for Coupled Nonlinear Schrödinger Equation

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Abstract. The numerical solution of CNLS equations are studied for periodic wave solutions. We use the first order partitioned average vector field method, the second order partitioned average vector field composition method and plus method. The nonlinear implicit schemes preserve the energy and the momentum. The results show that the methods are successful to get approximation.

Introduction

CNLS equations are presented by

$$\begin{aligned} &iu_t + \widetilde{\alpha}_1 u_{xx} + \left(\widetilde{\sigma}_1 |u|^2 + \widetilde{\mu} |v|^2\right) u = 0 \\ &iv_t + \widetilde{\alpha}_2 v_{xx} + \left(\widetilde{\sigma}_2 |v|^2 + \widetilde{\mu} |u|^2\right) v = 0 \end{aligned} \tag{0.1}$$

where $u(x,t)$ and $v(x,t)$ are the complex two dependent variables, x is the space variable and t is the time variable. The disperison coefficients are $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ [13, 15]. The CNLS system is included in many different area [8, 9, 12, 14, 7, 3].

We rarely get analytical solutions, like the Manakov model [8] for $\widetilde{\alpha}_1 = \widetilde{\alpha}_2 = 1, \quad \widetilde{\sigma}_1 = \widetilde{\sigma}_2 = \widetilde{\mu}$. Moreover, another cases are $\widetilde{\alpha}_1 = -\widetilde{\alpha}_2, \quad \widetilde{\sigma}_1 = \widetilde{\sigma}_2 = -\widetilde{\mu}$ that can be solved [16].

As a result of the interaction of wave packets in the system, nonlinear situations arise. In non-integrable cases, numerical methods should be used to analyze nonlinear situations.
Symplectic and multisymplectic method are used to solve CNLS equations and showed that both method preserve mass, energy and momentum in longtime evolution [1]. For soliton solution, Dehghan and Taleei examined a Chebyshev pseudospectral multidomain method. In [20], higher-order compact splitting multisymplectic method which is unconditionally stable are developed for solving CNLS equation. In [11], Variational iteration method was studied for solving CNLS equation . Ismail used Galerking method and tested this method for stabilty and accuracy [6]. The linearized conservative difference scheme was studied to solve the system [18]. In [19], the variational iteration method was applied to get soliton solutions for CNLS.
Energy preserving methods for solving PDE and ODE have been paid much attention nowadays. In [5, 4],the AVF methods of arbitrarily higher order were studied in which the Hamiltonian is canonical system and non-canonical system.

The authors in [17] construct a partitioned AVF method for the numerical solution of KGS system. If it is compare with the AVF method, according to concrete expression the last schemes are much easier. It remarkably improve the compuatational efficiency. The first method is first-order accuracy, but the other methods (PAVF-C, PAVF-P) are second-order accuracy [17]. In this paper, we apply

more efficient AVF based method to get numerical solution of CNLS. Up to the author's knowledge, a PAVF method for the CNLS equation (0.1) is never studied before.
This work given as follows. In Section 2, we propese four methods. Their application of CNLS equation given in Section 3. In Section 4, some conclusions are performed to give the competency and reliability of the presented methods in long term. Section 5 is included to concluding remarks.

1. PAVF Method

In this section, we will propese the partitioned average vector field model for the CNLS equation (0.1). Firstly, we give a second-order AVF method.

We consider PDEs with functions $y(\bar{x},\bar{t}) \in \mathfrak{R}$.

$$\frac{\partial y}{\partial \bar{t}} = S\frac{\delta \mathcal{H}}{\delta y} \tag{1.1}$$

where $(\bar{x},\bar{t}) \in \mathfrak{R}\times\mathfrak{R}$ is independent variables , S is a constant linear operator, $\frac{\delta\mathcal{H}}{\delta y}$ is variational derivative of $\mathcal{H}$

$$\mathcal{H}\left[y\right] = \int_{\eta} \tilde{H}(x,y^n)dx \tag{1.2}$$

where η is a subset of $\mathfrak{R}\times\mathfrak{R}$. System (1.1) can be rewrite using skew-gradient

$$\frac{\partial y}{\partial \bar{t}} = \bar{S}\nabla\bar{\mathcal{H}}(y), \quad \bar{S}^T = -\bar{S} \tag{1.3}$$

where $\bar{S}$ is now skew-symmetric matrix. We choose $\bar{\mathcal{H}}$ such that $\bar{\mathcal{H}}\triangle x$ is an estimation to $\mathcal{H}$. The variational derivate of $\mathcal{H}$ is dedicated by $\triangle\bar{\mathcal{H}}$. AVF method is defined for (1.3)

$$\frac{y_n^{j+1}-y_n^j}{\Delta\bar{t}} = \bar{S}\int_0^1 \nabla\bar{\mathcal{H}}((1-\varepsilon)y_n^j+\varepsilon y_n^{j+1})d\varepsilon \tag{1.4}$$

where the point y_n^j is the numerical value of $y(b+n\Delta x,\bar{t}_0+j\Delta\bar{t})$ for $\bar{x} \in [b,\ a]$, $\bar{t}_0$ is the initial time.
If $\bar{S}$ is skew-symmetric matrix such that ıt is approximation to S, then the method exactly preserve the energy.
The AVF method is time-symmetric [5]. Higher order linear integral pre- serving AVF methods, constructed using Gaussian quadrature for canonical and non-canonical Hamiltonian systems have even order 2s [4, 5]. The AVF method is connected to gradient methods [10].

For the partitioned AVF method, we consider the Hamiltonian system [17],

$$\dot{y} = \bar{S}\nabla\bar{\mathcal{H}} \tag{1.5}$$

Lets choose $y=(w,\bar{z})^T = (y_1,y_2,\ldots,y_m;y_{m+1},y_{m+2},\ldots,y_d)^T$, where d is an even number, denoting $d=2m$. Therefore, the system (1.5) can be rewritten as

$$\begin{pmatrix} \dot{w} \\ \dot{\bar{z}} \end{pmatrix} = \bar{S}_{2d}\begin{pmatrix} \bar{\mathcal{H}}_w(w,\bar{z}) \\ \bar{\mathcal{H}}_{\bar{z}}(w,\bar{z}) \end{pmatrix}, \quad w,\bar{z}\in\mathbb{R}^d \tag{1.6}$$

The Hamiltonian is conserved,

$$\frac{d\bar{\mathcal{H}}(w(t),\bar{z}(t))}{d\bar{t}} = 0 \tag{1.7}$$

Then the partitioned AVF method is given by

$$\frac{1}{\chi}\begin{pmatrix} w^{n+1}-w^n \\ \bar{z}^{n+1}-\bar{z}^n \end{pmatrix} = S_{2d}\begin{pmatrix} \int_0^1 \bar{\mathcal{H}}_w\left(\varepsilon w^{n+1}+(1-\varepsilon)w^n,\bar{z}^n\right)\mathrm{d}\varepsilon \\ \int_0^1 \bar{\mathcal{H}}_{\bar{z}}\left(w^{n+1},\varepsilon\bar{z}^{n+1}+(1-\varepsilon)\bar{z}^n\right)\mathrm{d}\varepsilon \end{pmatrix} \tag{1.8}$$

where χ is the time step.
In [17],the Hamiltonian of the system is preserved by the PAVF method exactly.

Remark 1. *[17] If we write $\bar{\mathcal{H}}(w, \bar{z}) = \bar{\mathcal{H}}_1(w) + \mathcal{H}_2(\bar{z})$ in (1.6) , then the method (1.8) is the same as AVF method (1.4). But, the system (1.6) is not separable generally.*

The adjoint of PAVF method (PAVF-ADJ) is defined by

$$\frac{1}{\chi}\begin{pmatrix} w^{n+1} - w^n \\ \bar{z}^{n+1} - \bar{z}^n \end{pmatrix} = S_{2d}\begin{pmatrix} \int_0^1 \bar{\mathcal{H}}_w\left(\varepsilon w^{n+1} + (1-\varepsilon)w^n, \bar{z}^{n+1}\right) \mathrm{d}\varepsilon \\ \int_0^1 \bar{\mathcal{H}}_{\bar{z}}\left(w^n, \varepsilon \bar{z}^{n+1} + (1-\varepsilon)\bar{z}^n\right) \mathrm{d}\varepsilon \end{pmatrix} \tag{1.9}$$

If we use the symbol $\Theta_{\Delta t}$ for the PAVF method (1.8), we can denote the PAVF-ADJ method as $\Theta^*_{\Delta t}$

Fınally, we define composition (PAVF-C) and plus method (PAVF-P) using PAVF method and PAVF-ADJ method. Composition method is defined by

$$\Phi_\chi := \Theta^*_{\frac{\chi}{2}} \circ \Theta_{\frac{\chi}{2}} \tag{1.10}$$

and plus method is given by

$$\hat{\Phi}_\chi := \frac{1}{2}\left(\Theta^*_\chi + \Theta_\chi\right) \tag{1.11}$$

2. Discretization of the system

Considering the equation (0.1), we can write the complex functions u, v of (0.1) sum of imaginary and real parts

$u(x, \bar{t}) = c(x, \bar{t}) + is(x, \bar{t}), \quad v(x, \bar{t}) = \tilde{c}(x, \bar{t}) + i\tilde{s}(x, \bar{t})$

the systems (0.1) is written

$$\begin{aligned} c_{\bar{t}} + \widetilde{\alpha}_1 s_{xx} + \left(\widetilde{\sigma}_1\left(c^2+s^2\right) + \widetilde{\mu}\left(\tilde{c}^2+\tilde{s}^2\right)\right) s = 0 \\ s_{\bar{t}} - \widetilde{\alpha}_1 c_{xx} - \left(\widetilde{\sigma}_1\left(c^2+s^2\right) + \widetilde{\mu}\left(\tilde{c}^2+\tilde{s}^2\right)\right) c = 0 \\ \tilde{c}_{\bar{t}} + \widetilde{\alpha}_2 \tilde{s}_{xx} + \left(\widetilde{\mu}\left(c^2+s^2\right) + \widetilde{\sigma}_2\left(\tilde{c}^2+\tilde{s}^2\right)\right) \tilde{s} = 0 \\ \tilde{s}_{\bar{t}} - \widetilde{\alpha}_2 \tilde{c}_{xx} - \left(\widetilde{\mu}\left(c^2+s^2\right) + \widetilde{\sigma}_2\left(\tilde{c}^2+\tilde{s}^2\right)\right) \tilde{c} = 0 \end{aligned} \tag{2.1}$$

These equations represent an infinite-dimensional Hamiltonian system in the phase space $\bar{z} = (c, s, \tilde{c}, \tilde{s})^T$

$$\bar{z}_t = \mathbf{S}^{-1}\frac{\delta\mathcal{H}}{\delta\bar{z}}$$

where $\mathbf{S}^{-1} = -\mathbf{S}$ and the Hamiltonian is

$$\mathcal{H} = \int_\Omega \left[\hat{w} - \frac{\widetilde{\alpha}_1}{2}\left(c_x^2 + s_x^2\right) - \frac{\widetilde{\alpha}_2}{2}\left(\tilde{c}_x^2 + \tilde{s}_x^2\right)\right] dx \tag{2.2}$$

with

$$\hat{w} = \frac{1}{4}\left(\widetilde{\sigma}_1\left(c^2+s^2\right)^2 + \widetilde{\sigma}_2\left(\tilde{c}^2+\tilde{s}^2\right)^2\right) + \frac{\widetilde{\mu}}{2}\left(c^2+s^2\right)\left(\tilde{c}^2+\tilde{s}^2\right)$$

Finite difference approximation is applied for the first-order derivatives (2.2) to get Hamiltonian system which is a finite-dimensional. From then, we obtain the discretized Hamiltonian (2.2)

$$\begin{aligned} \overline{\mathcal{H}} = & \sum_{r=1}^{N-1}\left\{\frac{1}{4}\left[\widetilde{\sigma}_1\left(c_r^2+s_r^2\right)^2 + \widetilde{\sigma}_2\left(\tilde{c}_r^2+\tilde{s}_r^2\right)^2\right] + \frac{\widetilde{\mu}}{2}\left(c_r^2+s_r^2\right)\left(\tilde{c}_r^2+\tilde{s}_r^2\right)\right\} \\ & - \sum_{r=1}^{N-1}\frac{\widetilde{\alpha}_1}{2}\left[\left(\frac{c_{r+1}-c_r}{\Delta x}\right)^2 + \left(\frac{s_{r+1}-s_r}{\Delta x}\right)^2\right] \\ & - \sum_{r=1}^{N-1}\frac{\widetilde{\alpha}_2}{2}\left[\left(\frac{\tilde{c}_{r+1}-\tilde{c}_r}{\Delta x}\right)^2 + \left(\frac{\tilde{s}_{r+1}-\tilde{s}_r}{\Delta x}\right)^2\right] \end{aligned} \tag{2.3}$$

Semi discrete Hamiltonian system is

$$\begin{pmatrix} c_{\bar{t}} \\ s_{\bar{t}} \\ \tilde{c}_{\bar{t}} \\ \tilde{s}_{\bar{t}} \end{pmatrix} = \bar{S}\nabla\bar{\mathcal{H}} = \begin{pmatrix} \frac{-\widetilde{\alpha}_1}{\Delta\bar{x}^2}\hat{A}s - \widetilde{\sigma}_1(c^2+s^2)s - \widetilde{\mu}(\tilde{c}^2+\tilde{s}^2)s \\ \frac{\widetilde{\alpha}_1}{\Delta\bar{x}^2}\hat{A}c + \widetilde{\sigma}_1(c^2+s^2)c + \widetilde{\mu}(\tilde{c}^2+\tilde{s}^2)c \\ \frac{-\widetilde{\alpha}_2}{\Delta\bar{x}^2}\hat{A}\tilde{s} - \widetilde{\sigma}_2(\tilde{c}^2+\tilde{s}^2)\tilde{s} - \widetilde{\mu}(c^2+s^2)\tilde{s} \\ \frac{\widetilde{\alpha}_2}{\Delta\bar{x}^2}\hat{A}\tilde{c} + \widetilde{\sigma}_2(\tilde{c}^2+\tilde{s}^2)\tilde{c} + \widetilde{\mu}(c^2+s^2)\tilde{c} \end{pmatrix} \tag{2.4}$$

with

$$\hat{A} = \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ 1 & & & & 1 & -2 \end{pmatrix}$$

The momentum conservation law is given with

$$\tilde{M}(\bar{z}) = \frac{1}{2}\left(c\frac{\partial s}{\partial \bar{x}} - s\frac{\partial c}{\partial \bar{x}} + \tilde{c}\frac{\partial \tilde{s}}{\partial \bar{x}} - \tilde{s}\frac{\partial \tilde{c}}{\partial \bar{x}}\right)$$

2.1. Derivation of the Method

We will apply the PAVF method for the CNLS equation (2.4)

$$\begin{aligned}
\frac{c^{n+1}-c^n}{\Delta\bar{t}} =& -\frac{\widetilde{\alpha}_1}{2}\hat{A}\left(s^n+s^{n+1}\right) \\
& -\frac{\widetilde{\sigma}_1}{4}\left[2\left(c^{n+1}\right)^2 s^n + 2\left(c^{n+1}\right)^2 s^{n+1} + \left(s^n\right)^3 + \left(s^n\right)^2 s^{n+1} + s^n\left(s^{n+1}\right)^2 + \left(s^{n+1}\right)^3\right] \\
& -\frac{\widetilde{\mu}}{2}\left[\left(\tilde{c}^n\right)^2 s^n + \left(\tilde{c}^n\right)^2 s^{n+1} + \left(\tilde{s}^n\right)^2 s^n + \left(\tilde{s}^n\right)^2 s^{n+1}\right] \\
\frac{s^{n+1}-s^n}{\Delta\bar{t}} =& \frac{\widetilde{\alpha}_1}{2}\hat{A}\left(c^n+c^{n+1}\right) \\
& +\frac{\widetilde{\sigma}_1}{4}\left[\left(c^n\right)^3 + \left(c^n\right)^2 c^{n+1} + c^n\left(c^{n+1}\right)^2 + \left(c^{n+1}\right)^3 + 2\left(s^n\right)^2 c^n + 2\left(s^n\right)^2 c^{n+1}\right] \\
& +\frac{\widetilde{\mu}}{2}\left[\left(\tilde{c}^n\right)^2 c^n + \left(\tilde{c}^n\right)^2 c^{n+1} + \left(\tilde{s}^n\right)^2 c^n + \left(\tilde{s}^n\right)^2 c^{n+1}\right] \\
\frac{\tilde{c}^{n+1}-\tilde{c}^n}{\Delta\bar{t}} =& -\frac{\widetilde{\alpha}_2}{2}\hat{A}\left(\tilde{s}^n+\tilde{s}^{n+1}\right) \\
& -\frac{\widetilde{\sigma}_2}{4}\left[2\left(\tilde{c}^{n+1}\right)^2 \tilde{s}^n + 2\left(\tilde{c}^{n+1}\right)^2 \tilde{s}^{n+1} + \left(\tilde{s}^n\right)^3 + \left(\tilde{s}^n\right)^2 \tilde{s}^{n+1} + \tilde{s}^n\left(\tilde{s}^{n+1}\right)^2 + \left(\tilde{s}^{n+1}\right)^3\right] \\
& -\frac{\widetilde{\mu}}{2}\left[\left(c^{n+1}\right)^2 \tilde{s}^n + \left(c^{n+1}\right)^2 \tilde{s}^{n+1} + \left(s^{n+1}\right)^2 \tilde{s}^n + \left(s^{n+1}\right)^2 \tilde{s}^{n+1}\right] \\
\frac{\tilde{s}^{n+1}-\tilde{s}^n}{\Delta\bar{t}} =& \frac{\widetilde{\alpha}_2}{2}\hat{A}\left(\tilde{c}^n+\tilde{c}^{n+1}\right) \\
& +\frac{\widetilde{\sigma}_2}{4}\left[\left(\tilde{c}^n\right)^3 + \left(\tilde{c}^n\right)^2 \tilde{c}^{n+1} + \tilde{c}^n\left(\tilde{c}^{n+1}\right)^2 + \left(\tilde{c}^{n+1}\right)^3 + 2\left(\tilde{s}^n\right)^2 \tilde{c}^n + 2\left(\tilde{s}^n\right)^2 \tilde{c}^{n+1}\right] \\
& +\frac{\widetilde{\mu}}{2}\left[\left(c^{n+1}\right)^2 \tilde{c}^n + \left(c^{n+1}\right)^2 \tilde{c}^{n+1} + \left(s^{n+1}\right)^2 \tilde{c}^n + \left(s^{n+1}\right)^2 \tilde{c}^{n+1}\right]
\end{aligned}$$

where

$$a_1 = \frac{\widetilde{\alpha}_1\Delta\bar{t}}{2}, \quad a_2 = \frac{\widetilde{\alpha}_2\Delta\bar{t}}{2}, \quad a_3 = \frac{\widetilde{\sigma}_1\Delta\bar{t}}{4}, \quad a_4 = \frac{\widetilde{\sigma}_2\Delta\bar{t}}{4}, \quad a_5 = \frac{\widetilde{\mu}\Delta\bar{t}}{2}$$

Notice that this system is nonlinear, we use Newton method to get the values $c^{n+1}, s^{n+1}, \tilde{c}^{n+1}, \tilde{s}^{n+1}$. So

Thereafter, derivation of the adjoint method, the composition method and the plus method for the CNLS equation be rewrite easly.
In the next section, we will get numerical experiments for the CNLS equation.

3. Numerical Results

In this part, we will apply all methods to the CNLS equation.
Using $N+1$ uniform grid points, the space interval $[b,a]$ is discretized. Where $\Delta x = h = \frac{a-b}{N}$ is grid spacing. Time interval is $0 \leq \bar{t} \leq T$. The system was solved by the four methods The global momentum and energy are defined by

$$G\tilde{M} = \Delta x \sum_{r=1}^{N} (\tilde{M}_r^n - \tilde{M}_i^0), \quad GE = \Delta x \sum_{r=1}^{N} (E_r^n - E_i^0) \tag{3.1}$$

where the initial energy is E^0 and the initial momentum is $\tilde{M}^0$. The accuracy of all methods checked at their conservation properties.
We use Newton method to solve system because for all methods we get the nonlinear equations.

3.1. Elliptic polarization

We use symmetric conditions

$$u(x,0) = 0.5(1 - 0.1\cos(0.5x)), \quad v(x,0) = u(x,0) \tag{3.2}$$

If the conditions are symmetric, the results are symmetric too (see [2]). $T = 100$ and the space interval is $[0, 8\pi]$. In the example, the solutions with periodic boundary conditions are considered.

Figure 1 and Figure 2 global errors of momentum and energy are ploted for all the four methods. The global error in energy conservation for PAVF-C method is better than others. For the energy, the global errors in are nearly likewise for three methods. We see that the energy, momentum do not expand with time for all methods.

Figure 3 and Figure 4 give the wave profiles of $|u|$ and $|v|$. It is clear that they are almost the same.

In Table 1, For the four methods, the CPU time is written . The CPU time of the PAVF-ADJ method is less than other methods.

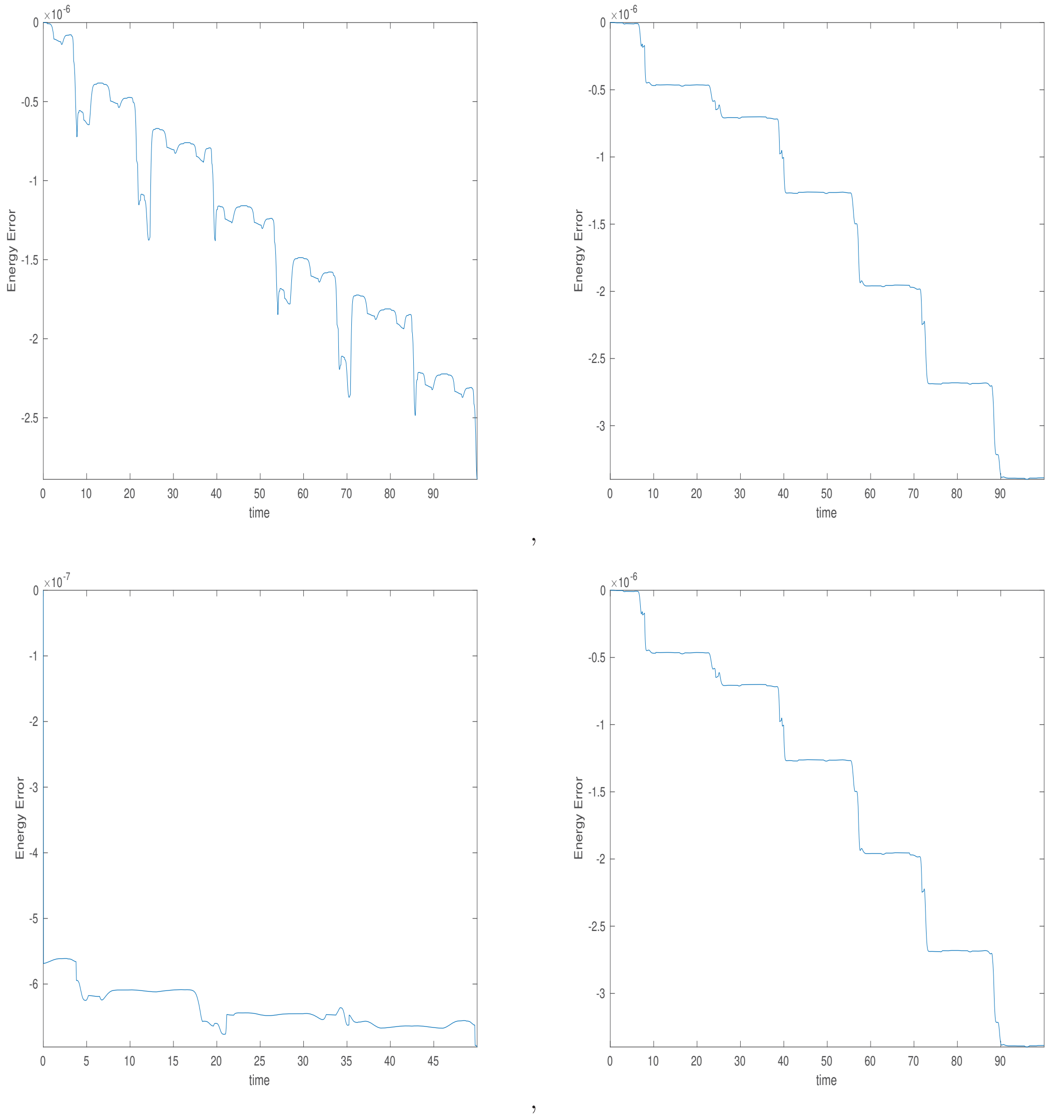

Figure 1. Global Enerji Error:PAVF method, PAVF-ADJ method, PAVF-C method, PAVF-P method

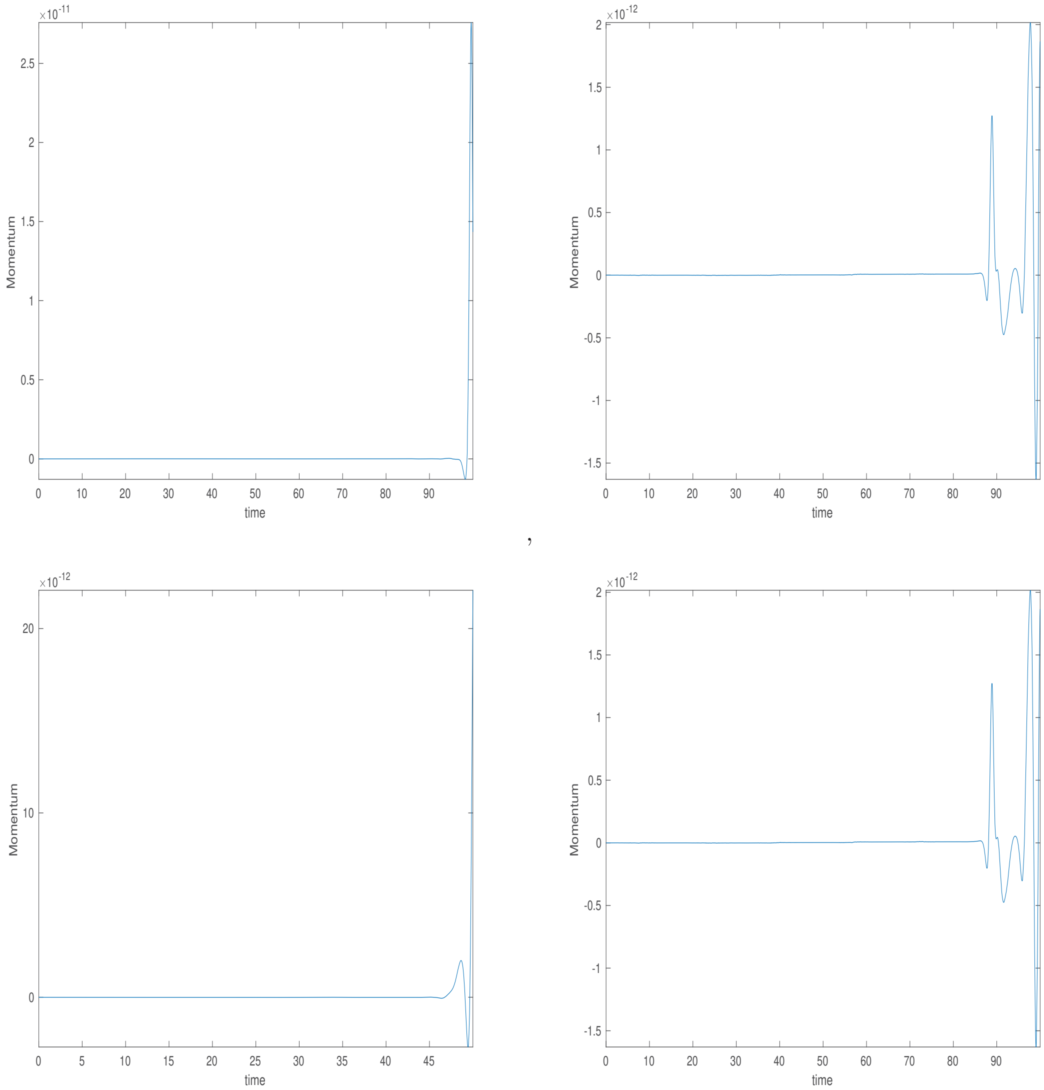

Figure 2. Global Momentum Error:PAVF method, PAVF-ADJ method,Composition method, Plus method

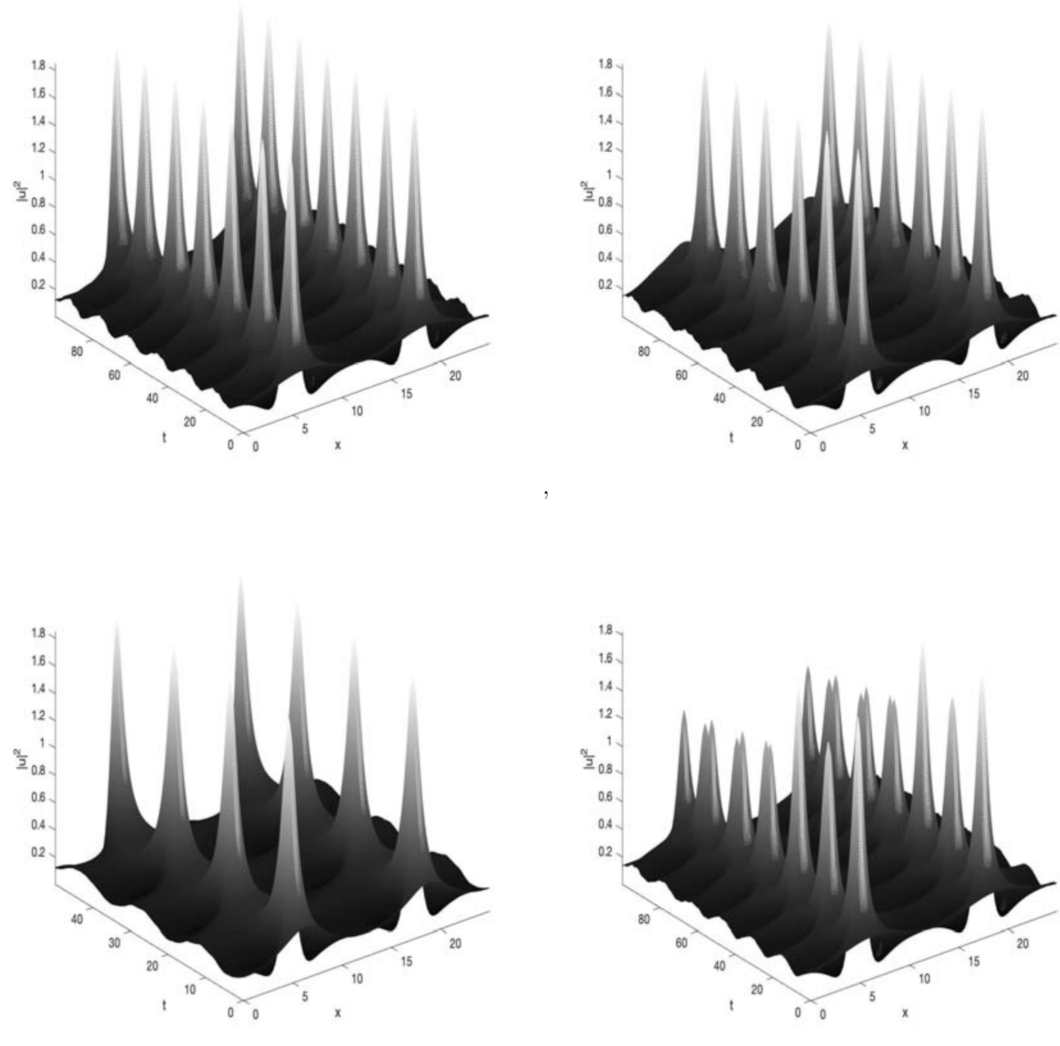

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Figure 3. Surface of v:PAVF method, PAVF-ADJ method,Composition method, Plus method

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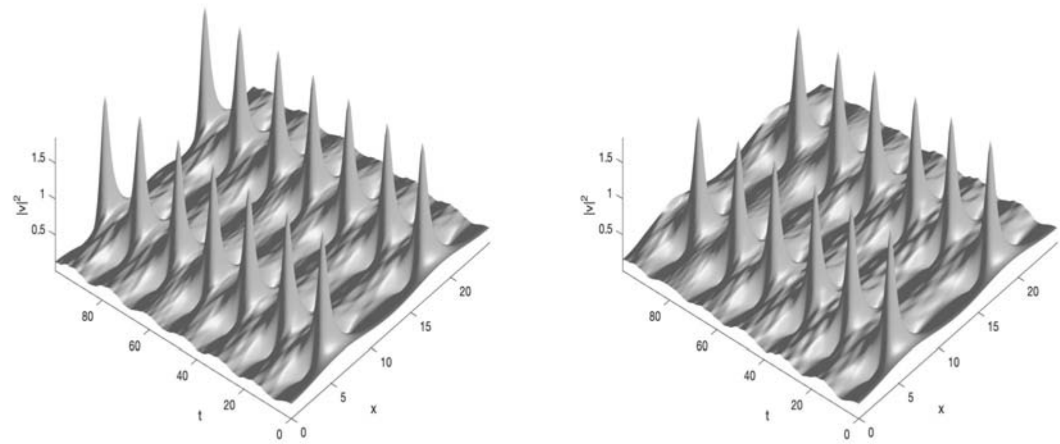

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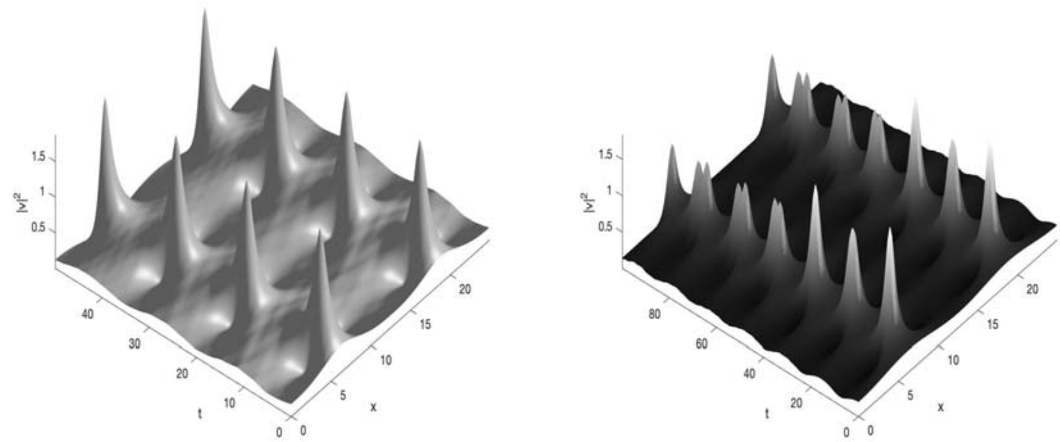

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Figure 4. Surface of v:PAVF method, PAVF-ADJ method, Composition method, Plus method

Table 1. For four energy-preserving methods, computational cost of one solitons

T=100	PAVF	PAVF-ADJ	PAVF-C	PAVF-P
CPU time	157.87	67.17	125.86	193.55

4. Conclusions

In this work, we give a PAVF, the adjoint of PAVF, PAVF Composition and PAVF Plus methods for CNLS equations with periodic solutions. All four methods preserve the momentum and energy in long time. Numerical experiments for a particular example are given. We compare four methods with CPU time. Numerical outcomes indicate that the given methods preserves momentum and energy exactly.

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