Energy Preserving Integration of The Strongly Coupled Nonlinear Schrödinger Equation

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Abstract. In this paper, average vector field method (AVF) is derived for strongly coupled Schrödinger equation (SCNLS). The SCNLS equation is discretized in space by finite differences and is solved in time by structure preserving AVF method. Numerical results for different paremeter compare with the Lobatto IIIA-IIIB method. The results indicate that AVF method are effective to preserve global energy and momentum.

Keywords: Strongly coupled nonlinear Schrödinger equation, Hamiltonian system, average vector field methods.
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INTRODUCTION

We consider the strongly coupled nonlinear Schrödinger (SCNLS) equation

$$\begin{aligned} iu_t + \beta u_{xx} + [\alpha_1|u|^2 + (\alpha_1 + 2\alpha_2)|v|^2]u + \gamma u + \Gamma v = 0 \\ iv_t + \beta v_{xx} + [\alpha_1|v|^2 + (\alpha_1 + 2\alpha_2)|u|^2]v + \gamma v + \Gamma u = 0 \end{aligned} \qquad (1)$$

with initial and boundary conditions

$$u(x,0) = u_0(x), v(x,0) = v_0(x), u(x_L,t) = u(x_R,t), v(x_L,t) = v(x_R,t), \quad t > 0 \qquad (2)$$

where $u(x,t)$ and $v(x,t)$ are complex-valued functions of the spatial coordinate x and time $t, \beta, \alpha_1, \alpha_2, \gamma, \Gamma$ are real constants. The parameter β describes the group velocity dispersion and the term proportional to α_1 describes the self-focusing of a signal for pulses in birefringent media.The nonlinear coupling or cross-modulation parameter $\alpha = \alpha_1 + 2\alpha_2$ desribes how each component of the solution is influenced by the other component. The constant γ is the ambient potential, called normalized birefringent. The parameter Γ is the linear coupling parameter, also called the linear birefringent. SCNLS equation has extensive application in many problems of mathematical physics, nonlinear optics, plasma physics, as well as biological structures. It was shown that the SCNLS equation is completely integrable for the Manakov case with $\alpha_2 = 0$. Also analytic solutions exists via inverse scattering transformation for the case with $\alpha_2 = \gamma = \Gamma$=0. In recent years, many research has

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been done for numerical solution of the SCNLS equation. Sonnier and Christov [1] have applied the conservative implicit Crank-Nicholson method. Todorrov and Christov [2] investigates numerically the role of linear and nonlinear coupling by taking $\Gamma = 0$. Finite difference scheme [3] and implicit Pressmann scheme [4] were used to get numerical solution of the SCNLS. Two stage Lobatto IIIA-IIIB method [5] is applied to the SCNLS equation.

Recently, much attention have paid to energy preserving methods for solving PDE and ODE. One of them is average vector field method (AVF) which is extention of the implicit-mid point rule. Higher order AVF methods are constructed by using the Guassian quadrature and they are interpreted as Runge-Kutta method with continuous stages. The AVF methods of arbitrarily higher order were developed and analyzed for canonical and non-canonical Hamiltonian systems in [6,7]. A relation between the energy preservation and symplecticness is establihed by so called B-series. The AVF method and its high order are B-series methods [6,7]. Using the B-series, it can shown that the AVF method for canonical and non-canonical Hamiltonian systems is conjugate to symplectic or Poisson integrators [6,7]. Application of the AVF method for various nonlinear evolutionary partial differential equation was given in [8]. Karasözen and Erdem investigated AVF method for the Volterra lattice equation as non-canonical Hamiltonian system [9].

The paper organized as follows. In section 2 AVF method is described briefly and applied to SCNLS equation. In section 3 we present numerical results for different parameter.

AVF METHOD for SCNLS

We consider evolutionary PDEs with independent variables $(x,t)\epsilon\mathbb{R}x\mathbb{R}$, functions $y(x,t)\epsilon\mathbb{R}$ and PDEs of the form

$$\frac{\partial y}{\partial t} = S\frac{\delta H}{\delta y} \tag{3}$$

where S is a constant linear differential operator, $\frac{\delta H}{\delta y}$ variational derivative of H. Conservative PDEs (3) can be semi-discretised in "skew-gradient" form

$$\frac{\partial y}{\partial t} = \bar{S}\nabla\bar{H}(y), \qquad \bar{S}^T = -\bar{S} \tag{4}$$

when $\bar{S}$ skew-adjoint. $\bar{H}$ is chosen in such a way that $\bar{H}\Delta x$ is an approximation to $\bar{H}$. The discrete analogue of the variational derivative $\frac{\delta H}{\delta y}$ is given by $\nabla\bar{H}$.

The average vector field (AVF) method is defined for (4)

$$\frac{y_n^{j+1} - y_n^j}{\Delta t} = \bar{S}\int_0^1 \nabla\bar{H}((1-\varepsilon)y_n^j + \varepsilon y_n^{j+1})\, d\varepsilon \tag{5}$$

where the point y_n^j is the discrete equivalent of $y(a+n\Delta x, t_0 + j\Delta t)$ for $x \in [a,b]$, t_0 is the initial time, Δx is spatial step lenght and Δt is time step lenght.

If $\bar{S}$ is skew-symmetric matrix approximation to S then the average vector field method exactly preserve the energy.

By decomposing the complex functions ψ_1, ψ_2 of (1) into real and imaginary parts

$$u = p + iq, \quad v = \mu + i\xi \tag{6}$$

the SCNLS systems (1) can be written as a system of real-valued equations

$$\begin{aligned} p_t + \beta q_{xx} + [\alpha_1(p^2+q^2) + \alpha(\mu^2+\xi^2)]q + \gamma q + \Gamma\xi = 0 \\ -q_t + \beta p_{xx} + [\alpha_1(p^2+q^2) + \alpha(\mu^2+\xi^2)]p + \gamma p + \Gamma\mu = 0 \\ \mu_t + \beta \xi_{xx} + [\alpha_1(\mu^2+\xi^2) + \alpha(p^2+q^2)]\xi + \gamma\xi + \Gamma q = 0 \\ -\xi_t + \beta\mu_{xx} + [\alpha_1(\mu^2+\xi^2) + \alpha(p^2+q^2)]\mu + \gamma\mu + \Gamma p \\ = 0 \end{aligned} \tag{7}$$

These equations represent an infinite-dimensional Hamiltonian system in the phase space $z = (p, q, \mu, \xi)$

$$z_t = S^{-1}\frac{\delta H}{\delta y},$$

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{8}$$

where the Hamiltonian is

$$H = \int_\Omega \left\{ W - \frac{\beta}{2}\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \mu}{\partial x}\right)^2\right]\right\} dx \tag{9}$$

$$\begin{aligned} W = \frac{\alpha_1}{4}[(p^2+q^2)^2 + (\mu^2+\xi^2)^2] + \frac{\alpha}{2}(p^2+q^2)(\mu^2+\xi^2) \\ + \frac{\gamma}{2}(p^2+q^2+\mu^2+\xi^2) + \Gamma(\mu p + \xi q) \end{aligned} \tag{10}$$

We take Ω for $[a, b]$. We use finite difference approximation for the first-order derivatives in the Hamiltonian to obtain a finite-dimensional Hamiltonian system. The discretized Hamiltonian is then

$$\begin{aligned} \bar{H} = \sum_{i=1}^{n-1} \frac{\alpha_1}{4}[(p_i^2+q_i^2)^2 + (\mu_i^2+\xi_i^2)^2] + \frac{\alpha}{2}(p_i^2+q_i^2)(\mu_i^2+\xi_i^2) \\ + \frac{\gamma}{2}(p_i^2+q_i^2+\mu_i^2+\xi_i^2) + \Gamma(\mu_i p_i + \xi_i q_i) - \frac{\beta}{2}\left[\left(\frac{q_{i+1}-q_i}{\Delta x}\right)^2\right. \\ \left. + \left(\frac{p_{i+1}-p_i}{\Delta x}\right)^2 + \left(\frac{\mu_{i+1}-\mu_i}{\Delta x}\right)^2 + \left(\frac{\xi_{i+1}-\xi_i}{\Delta x}\right)^2\right] \end{aligned} \tag{11}$$

Semi discrete Hamiltonian system is

$$p_t = -\alpha_1 q(p^2+q^2) - \alpha q(\mu^2+\xi^2) - \gamma q - \Gamma\xi + \beta A q$$
$$q_t = \alpha_1 p(p^2+q^2) + \alpha p(\mu^2+\xi^2) + \gamma q + \Gamma\mu - \beta A p$$
$$\mu_t = -\alpha_1 \xi(\mu^2+\xi^2) - \alpha\xi(p^2+q^2) - \gamma\xi - \Gamma q + \beta A\xi$$
$$\xi_t = \alpha_1\mu(\mu^2+\xi^2) - \alpha\mu(p^2+q^2) + \gamma\mu + \Gamma p - \beta A\mu \qquad (12)$$

For the discretization of the time derivative of system (12), finite differences is used and for left side of system we took integral from 0 to 1, i.e. we apply AVF method to semi discrete Hamiltonian system (12). The integral in the AVF method (5) can be calculated exactly. We use Newton method to solve the nonlinear system.

In addition to, under periodic boundary conditions we have also momentum conservation. Local momentum conservation law is

$$I(z) = \frac{1}{2}(qp_x + \xi - pq_x - \mu\xi_x) \qquad (13)$$

NUMERICAL RESULTS

In order to investigate the performance of the AVF method developed in Section 2, in numerical calculation we choose non-zero values for parameter α_2, γ and Γ. In all numerical examples we took the $\beta = \alpha_1 = \gamma = 1.0$. We got elastic collisions for the parameters $\alpha_2 = -\frac{1}{6}, \Gamma = 1.0$ and inelastic collisions for $\alpha_2 = -\frac{1}{6}, \Gamma = 0.0175$. The choice of the parameters $\alpha_2 = -1/3, \Gamma = 0.0175$ resulted in the fusion of the two solitions. The space interval
$[x_L, x_R]$ is discretized by $N+1$ uniform grid points with grid spacing spacing $\Delta x = h = (x_R - x_L)/N$. We compute the solution for the time interval $0 \le t \le T$.
The global energy error is given

$$GE = \Delta x \sum_{j=1}^{N} \left(E_j^n - E_j^0\right) \qquad (14)$$

where E^0 is the initial energy. Global error in momentum and norm conservation laws can be define analogously.
First, we consider the elastic collision, the inelastic collision of two solitions and the fusion of two solitons by taking as the initial conditions

$$u(0,x) = \sqrt{2}\,\mathrm{sech}\left(x + \frac{1}{2}D_0\right)\exp\left(\frac{iV_0 x}{4}\right) \qquad (15)$$

$$v(0,x) = \sqrt{2}\,\mathrm{sech}\left(x - \frac{1}{2}D_0\right)\exp\left(-\frac{iV_0 x}{4}\right) \qquad (16)$$

with $D_0 = 25, V_{0=1.0}$. For the elastic collision and the inelastic collision of two solitions we take $T = 50, dt = 0.01, dx = \frac{60}{128}, x_L = -30, x_R = 30$. And we use $T = 80, dt = 0.01, dx = \frac{60}{128}, x_L = -30, x_R = 30$ for the fusion of two solitons.

Finally, we consider the periodic solution of the SCNLS equation by choosing the parameters as $\beta = \alpha_1 = \gamma = 1.0, \alpha_2 = -\frac{1}{6}, \Gamma = 0.175$ and taking as the initial condition

$$u(0,x) = a_0(1 - \xi\cos(lx)), \qquad v(0,x) = b_0(1 - \xi\cos(lx)) \tag{17}$$

with $a_0 = 0.5, b_0 = 0.5, \xi = 0.1, l = 0.5$.

For the periodic solution we take $T = 100, dt = 0.01, dx = \frac{12}{128}, x_R = 6, x_L = -6$. Table (1) represents the global energy for elastic collision, inelastic collision, the fusion of two solitons and periodic solution. We see that AVF method is preserved the energy better than Lobatto IIIA-IIIB method in long time integration. Table (2) shows that the global momentum is preserved more accurately. In the case of periodic solutions Lobatto IIIA-IIIB method is preserved the global momentum more accurately than in case of soliton solutions.

TABLE (1). Errors in energy conservation for SCNLS.

	Lobatto IIIA-IIIB	**AVF Method**
Elastic collision	-0.3E-1	2.5E-14
Inelastic collision	-0.4E-1	3.5E-14
Fusion of solutions	-0.3E-1	0.6E-13
Periodic solutions	0.1E-3	2.5E-14

TABLE (2). Errors in momentum conservation for SCNLS.

	Lobatto IIIA-IIIB	**AVF Method**
Elastic collision	0.4E-2	1.5E-9
Inelastic collision	0.2E-2	2E-9
Fusion of solutions	0.5E-1	2.5E-8
Periodic solutions	0.4E-13	0.1E-4

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