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Some Result For The Janowski's P-Fold Symmetric Starlike Functions Of Complex Order

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Abstract. Janowski's starlike functions of complex order and have power series of the form

$$f(z) = z + a_{p+1}z^{p+1} + a_{2p+1}z^{2p+1} + \dots$$

where $p = 1,2,3,\dots$ are shown to satisfy the relation $f(z) = [g(z^p)]^{1/p}$ where $g(z)$ is Janowski's starlike functions of complex order with power series $g(z) = z + b_2z^2 + b_3z^3 + \dots$. Distortion results and Koebe domains are obtained.

Özet. $f(z) = z + a_{p+1}z^{p+1} + a_{2p+1}z^{2p+1} + \dots$, $p = 1,2,3,\dots$ açılımına sahip kompleks mertebeden Janowski yıldızıl fonksiyonlarının $g(z) = z + b_2z^2 + \dots$ açılımına sahip kompleks mertebeden Janowski yıldızıl fonksiyonlar ile $f(z) = [g(z^p)]^{1/p}$ bağıntısını gerçekledikleri gösterilmiştir. Söz konusu sınıf için distorsiyon teoremi ve Koebe bölgeleri bulunmuştur.

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1. Introduction. In a recent paper [4] Y.Polatoğlu and M.Bolcal obtained distortion theorems, Koebe domain and the radius of starlikeness for the class of Janowski's starlike functions of complex order. In this paper we look at functions which are Janowski's p-fold symmetric starlike functions of complex order. Specifically we look at functions f which are Janowski's starlike functions of complex order with power series of the form

$$f(z) = z + a_{p+1}z^{p+1} + a_{2p+1}z^{2p+1} + \dots \tag{1.1}$$

where $p = 1,2,3,\dots$

For completeness we recall the pertinent definitions and theorems.

Definition.1.1. Let Ω be the family of functions $\omega(z)$ regular in the disc $D = \{z \mid |z| < 1\}$ and satisfying the condition $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in D$.

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Next, for arbitrary fixed numbers A, B, $-1 < A \leq 1$, $-1 \leq B < A$, denote by $P(A,B)$ the family of functions

$$(1.2) \qquad p(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

regular in D and such that $p(z) \in P(A,B)$ if and only if

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

for some functions $\omega(z) \in \Omega$ and every $z \in D$.

Morever, Let $S_1^*(A,B,b)$, $(b \neq 0, \text{Complex})$ denote the family of functions

$$(1.3) \qquad g(z) = z + b_2 z^2 + \ldots$$

regular in D and such that $g(z)$ is in $S_1^*(A,B,b)$, $(b \neq 0, \text{Complex})$ if and only if

$$(1.4) \qquad \left[1 + \frac{1}{b}\left(z \cdot \frac{g'(z)}{g(z)} - 1\right)\right] = p(z)$$

for some $p(z)$ in $P(A,B)$ and all z in D. The class $S_1^*(A,B,b)$ is the Janowski's starlike functions of complex order.

Definition 1.2. If $f(z) \in S_1^*(A,B,b)$ and has a power series of the form (1.1) we write $f(z) \in S_p^*(A,B,b)$.

One of the results of this paper will depend upon the theorem (proven in II) that $f(z) \in S_p^*(A,B,b)$ iff $g(z) \in S_1^*(A,B,b)$ where $f(z) = \left[g(z^p)\right]^{1/p}$. The subsequent distortion theorems and Koebe domain. (Proven in II and III respectively) will follow from results in [4]

II. The Basic Relation: In this section we consider the following result.

Theorem 2.1. $f(z) \in S_p^*(A,B,b)$ iff $g(z) \in S_1^*(A,B,b)$, where

$$f(z) = \left[g(z^p)\right]^{1/p}$$

Proof: Let $f(z)\in S_p(A,B,b)$, thus

$$\left[1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)\right]=p(z),\quad p(z)\in P(A,B) \tag{2.1}$$

setting $f(z)=\left[g(z^p)\right]^{1/p}$ and computing $\left[1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)\right]$

$$\left[1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)\right]=\left[1+\frac{1}{b}\left(z^p\frac{g'(z^p)}{g(z^p)}-1\right)\right] \tag{2.2}$$

is obtained. We notice that the left-hand side of (2.1) is equal to

$$\left[1+\frac{1}{b}\left(z^p\frac{g'(z^p)}{g(z^p)}-1\right)\right]$$

but the condition that this quantity is equal to

$$\left[1+\frac{1}{b}\left(z^p\frac{g'(z^p)}{g(z^p)}-1\right)\right]=p(z)\ ,\ p(z)\in P(A,B)$$

and the condition that is equivalent to $g(z)\in S_p^*(A,B,b)$. Since the computations are reversible it follows that $f(z)\in S_p(A,B,b)$ iff $g(z)\in S_1(A,B,b)$. The proof of this theorem is based on α method introduced by. H.B.Coonce and S.S.Miller. [2].

Theorem.2.2 If $f(z)$ α Janowski p-fold symmetric starlike function of complex order, the for $|z|=r<1$

$$\left[F(r^p;-A,-B,|b|)\right]^{1/p}\le|f(z)|\le\left[F(r^p;A,B,|b|)\right]^{1/p} \tag{2.3}$$

where

$$\left[F(r^p;A,B,|b|)\right]^{1/p}=\begin{cases} r(1+Br^p)^{\frac{|b|(A-B)}{Bp}} & \text{if } B\neq 0\\ re^{\frac{|b|Ar}{p}} & \text{if } B=0\end{cases}$$

This bound is sharp. Because the extremal function is

(2.4)
$$f_*(z)=\begin{cases} z(1+Bz^p)^{\frac{b(A-B)}{Bp}} & \text{if } B\neq 0\\ ze^{\frac{bAz}{p}} & \text{if } B=0\end{cases}$$

Proof: In [4] it is shown that for $g(z)\in S_1^*(A,B,b)$,

(2.5)
$$F(r;-A,-B,|b|)\leq |g(z)|\leq F(r;A,B,|b|)$$

By theorem 2.1 $f(z)=[g(z^p)]^{\frac{1}{p}}$ and (2.3) follows. Since (2.5) is sharp for

$$f_{**}(z)=\begin{cases} z(1+Bz)^{\frac{b(A-B)}{B}} & \text{if } B\neq 0\\ ze^{bAz} & \text{if } B=0\end{cases}$$

we have equality for $f_*(z)$.

Corollaries:

1. For $A=1$, $B=-1$, $\frac{r}{(1+r^p)^{\frac{2|b|}{p}}}\leq |f(z)|\leq \frac{r}{(1-r^p)^{\frac{2|b|}{p}}}$

In this Case;

(i) For $b=1$, $\frac{r}{(1+r^p)^{\frac{2}{p}}}\leq |f(z)|\leq \frac{r}{(1-r^p)^{\frac{2}{p}}}$

This result is the well known, which was obtained by M.S.Robertson [3]. On the other hand, if we take $p=1$ and $p=2$ we obtain

(2.6)
$$\frac{r}{(1+r)^2}\leq |f(z)|\leq \frac{r}{(1-r)^2}$$

(2.7) $$\frac{r}{1+r^2} \le |f(z)| \le \frac{r}{1-r^2}$$

(2.6) is the distortion for the class of starlike functions which is the well known result.[1]. (2.7) is the distortion for all odd starlike functions [1].

(ii) $b = 1-\alpha$, $0 \le \alpha < 1$

$$\frac{r}{\left(1+r^p\right)^{\frac{2(1-\alpha)}{p}}} \le |f(z)| \le \frac{r}{\left(1-r^p\right)^{\frac{2(1-\alpha)}{p}}}$$

This is the distortion for the class of p-fold symmetric starlike functions of order α. In this case if we take $p=1$ and $p=2$

(2.8) $$\frac{r}{(1+r)^{2(1-\alpha)}} \le |f(z)| \le \frac{r}{(1-r)^{2(1-\alpha)}}.$$

This result is the well known, which was obtained by.M.S.Robertson. [3]

(2.9) $$\frac{r}{\left(1+r^2\right)^{(1-\alpha)}} \le |f(z)| \le \frac{r}{\left(1-r^2\right)^{(1-\alpha)}}$$

This is the distortion for the class of odd starlike functions of order α.

(iii) $b = e^{-i\lambda}\cos\lambda$, $|\lambda| < \frac{\pi}{2}$. $\frac{r}{\left(1+r^p\right)^{\frac{2\lambda}{p}}} \le |f(z)| \le \frac{r}{\left(1-r^p\right)^{\frac{2\cos\lambda}{p}}}$

This is the distortion for the class of p-fold symmetric α-spirallike functions.
In this case if we take $p=1$ and $p=2$ we get

(2.10) $$\frac{r}{(1+r)^{2\cos\lambda}} \le |f(z)| \le \frac{r}{(1-r)^{2\cos\lambda}}$$

This is the distortion for the class of α-spirallike functions, which was obtained by Y.Polatoğlu and M.Bolcal. [4].

(2.11) $$\frac{r}{\left(1+r^2\right)^{\cos\lambda}} \le |f(z)| \le \frac{r}{\left(1-r^2\right)^{\cos\lambda}}$$

This is the distortion for the class of odd λ-spirallike functions

(iv) $b=(1-\alpha)e^{i\lambda}\cos\lambda,\ 0\le\alpha<1,\ |\lambda|<\frac{\pi}{2}$

$$\frac{r}{\left(1+r^p\right)^{\frac{2(1-\alpha)\cos\lambda}{p}}}\le|f(z)|\le\frac{r}{\left(1-r^p\right)^{\frac{2(1-\alpha)\cos\lambda}{p}}}$$

This is the distortion for the class of p-fold symmetric λ-spirallike functions of order α.

(2.12) for $p=1$ $$\frac{r}{(1+r)^{2(1-\alpha)\cos\lambda}}\le|f(z)|\le\frac{r}{(1-r)^{2(1-\alpha)\cos\lambda}},$$

(2.13) for $p=2$ $$\frac{r}{\left(1+r^2\right)^{(1-\alpha)\cos\lambda}}\le|f(z)|\le\frac{r}{\left(1-r^2\right)^{(1-\alpha)\cos\lambda}},$$

The inequality (2.12) is the distortion for the class of λ-spirallike functions of order α and inequality (2.13) is the distortion for the class of odd λ-spirallike functions of order α.

2. For $A=1-2\beta,\ B=1,\ 0<\beta<1$

$$\frac{r}{\left(1+r^p\right)^{\frac{2|b|(1-\beta)}{p}}}\le|f(z)|\le\frac{r}{\left(1-r^p\right)^{\frac{2|b|(1-\beta)}{p}}}$$

This is the distortion for the class of $S_p^*(1-2\beta,-1,b)$. This class is the set defined by

$$\mathrm{Re}\left[1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)\right]>\beta.$$

In this case if we give specific values to b, we obtain that the distortions for the corresponding classes.

3. For $A=1,\ B=0$,

$$re^{-\frac{|b|}{p}r}\le|f(z)|\le re^{\frac{|b|}{p}r}.$$

This is the distortion for the class $S_p^*(1,0,b)$. This class is the set defined by

$$\left|\left[1-\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)\right]-1\right|<1.$$

It should be noticed that by giving the specific values to b, we obtain the distortions for the corresponding classes.

4. For $A=\beta$, $B=0$, $0<\beta<1$

$$re^{-\frac{|b|\beta}{p}r}\le|f(z)|\le re^{\frac{|b|\beta}{p}r}$$

This inequality is the distortion for the class $S_p^*(\beta,0,b)$. Similary if we give special values to b, we obtain the distortions for the classes $S_p^*(\beta,0,1)$, $S_p^*(\beta,0,1-\alpha)$, $0\le\alpha<1$, $S_p^*(\beta,0,e^{-i\lambda}\cos\lambda)$, $|\lambda|<\frac{\pi}{2}$ and $S_p^*(\beta,0,(1-\alpha)e^{-i\lambda}\cos\lambda)$, $0\le\alpha<1$, $|\lambda|<\frac{\pi}{2}$.

5. For $A=\beta$, $B=-\beta$, $0<\beta<1$

$$\frac{r}{(1+\beta r^p)^{\frac{2|b|}{p}}}\le|f(z)|\le\frac{r}{(1-\beta r^p)^{\frac{2|b|}{p}}}$$

This inequality is the distortion for the class $S_p^*(\beta,-\beta,b)$ which is defined by

$$\left|\frac{S(f(z),b)-1}{S(f(z),b)+1}\right|<\beta$$

where

$$S(f(z),b)=1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right).$$

We note that giving by special values to b, we obtain the distortions for the classes $S_p^*(\beta,-\beta,1)$, $S_p^*(\beta,-\beta,1-\alpha)$, $0\le\alpha<1$, $S_p^*(\beta,-\beta,e^{-i\lambda}\cos\lambda)$, $|\lambda|<\frac{\pi}{2}$ and $S_p^*(\beta,-\beta,(1-\alpha)e^{-i\lambda}\cos\lambda)$, $0\le\alpha<1$, $|\lambda|<\frac{\pi}{2}$.

6. For $A=1$, $B=-1+\frac{1}{M}$, $M>\frac{1}{2}$.

$$r\left[1-\left(\frac{1}{M}-1\right)r^{p}\right]^{\frac{|b|\left(2-\frac{1}{M}\right)}{p\left(\frac{1}{M}-1\right)}} \le |f(z)| \le r\left[1+\left(\frac{1}{M}-1\right)r^{p}\right]^{\frac{|b|\left(2-\frac{1}{M}\right)}{p\left(\frac{1}{M}-1\right)}}$$

This inequality is the distortion for the class $S_p^*\left(1,-1+\frac{1}{M},b\right)$ which is the set defined by

$$\left|\left[1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)\right]-M\right| < M$$

In this case if we give the specific values to b we obtain the distortions for the corresponding classes.

III. Koebe Domains: In this section we shall give the Koebe domain for the classes of univalent functions which are contains in $S_p^*(A,B,b)$.

From the definition Koebe domain [See 1. page 113].

$$R=\lim_{r\to 1}\left[r\left(1-Br^{p}\right)^{\frac{|b|(A-B)}{Bp}}\right]=\left[\left(1-B^{\frac{|b|(A-B)}{Bp}}\right)\right] \text{ if } B\neq 0$$

1) For $A=1$, $B=-1$, $R=\frac{1}{2^{\frac{2|b|}{p}}}$

In this case;

i) For $b=1$. $R=\frac{1}{4^{1/p}}$. This is the well known on result, which was obtained by H.B.Coonce and S.S.Miller [2]. If we take $p=1$ we obtain $R=\frac{1}{4}$ is the Koebe domain for the class of starlike functions, which is the well known result [See. 1. page. 114],

(ii) For $b=1-\alpha$, $R=\frac{1}{4^{\frac{1-\alpha}{p}}}$. This is the Koebe domain for the class of p-fold symmetric starlike functions of order α. In this case, for $p=1$ we obtain $R=\frac{1}{4^{1-\alpha}}$. This is the Koebe domain for the class of starlike functions of order α, which is the well known result [See 1. page 114]. If we take $p = 2$ we get $R = \frac{1}{4^{\frac{1-\alpha}{2}}}$ is the Koebe domain for the class of odd starlike functions of order α.

(iii) For $b = e^{-i\lambda}\cos\lambda$, $|\lambda| < \frac{\pi}{2}$, $R = \frac{1}{4^{\frac{\cos\lambda}{p}}}$ is the Koebe domain for the class of p-fold symmetric λ-spirallike functions. For $p = 2$, $R = \frac{1}{4^{\frac{\cos\lambda}{2}}}$ is the Koebe domain for the class of odd λ-spirallike functions.

(iv) For $b = (1-\alpha)e^{-i\lambda}\cos\lambda$, $0 \le \alpha < 1$, $|\lambda| < \frac{\pi}{2}$,

$R = \frac{1}{4^{\frac{(1-\alpha)\cos\lambda}{p}}}$ is the Koebe domain for the class of p-fold symmetric λ-spirallike functions of order α. If we take $p = 2$ we obtain $R = \frac{1}{4^{\frac{(1-\alpha)\cos\lambda}{2}}}$ is the Koebe domain for the class of odd λ – spirallike functions of order α.

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