

A Coefficient Inequality for Convex Functions

Yaşar Polatoğlu *

Abstract

In this study an important result of the paper called 'A characterization for convex functions of complex order'(İst. Üniv. Fen Fak. Matematik Dergisi cilt 54 sayfa 175- 179, 1997) is given and we present a coefficient inequality for convex functions under the regularly univalent conditions.

Özet

Biz bu makalede 'A characterization for convex functions of complex order (İst. Üniv. Fen Fak. Matematik Dergisi cilt 54 sayfa 175-170, 1997) adlı makalenin çok önemli bir neticesi olan katsayı eşitsizliğini veririz.

Keywords : Coefficient inequality, λ -Spirallike functions, Convex function of complex order.

Introduction:

Let R denote the class of functions

$$f(z) = z + a_2z^2 + a_3z^3 + \dots$$

which are analytic in the unit disc $D = \{z / |z| \leq 1\}$

A function $f(z)$ in R , is said to be a convex function of complex order b ($b \neq 0$, complex) that is $f(z) \in C(b)$ if and only if $f'(z) \neq 0$, and

$$\operatorname{Re} \left(1 + \frac{1}{b} z \cdot \frac{f''(z)}{f'(z)} \right) \geq 0, z \in D$$

The class $C(b)$ was introduced by P. Wiatrowski [3]. By giving specific values to b , we obtain the following important subclasses:

- (i) $C(1)$ is a well known class of convex functions,
- (ii) $C(1 - \beta)$, $0 \leq \beta \leq 1$ is the class of convex functions of order β ,
- (iii) $C(e^{-i\lambda} \cdot \cos \lambda)$, $|\lambda| \leq \frac{\pi}{2}$ is the class of functions for which $z \cdot f'(z)$ is λ Spirallike,
- (iv) $C((1 - \beta) \cdot e^{-i\lambda} \cdot \cos \lambda)$, $0 \leq \beta < 1$, $|\lambda| < \frac{\pi}{2}$ is the class of functions for which $z \cdot f'(z)$ is λ - Spirallike of order β [See. 1,3,4,5].

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Theorem 1.1.

Let

$$f(z) = z + a_2z^2 + a_3z^3 + \dots$$

be analytic in D. A necessary and sufficient condition that

$$f(z) \in C(b)$$

is for each real number $k, \dots -1 < k < 1$, the functions $F(k, b, z, \eta)$ defined by the equations, is

$$(1.2) \quad F(k, b, z, \eta) = \left[\frac{k(f(z) - f(\eta))}{f(kz) - f(k\eta)} \right]^{\frac{1}{b}}$$

$$(1.3) \quad F(k, b, 0, 0) = 1$$

$$(1.4) \quad F(1, b, z, \eta) = \left[\frac{f(z) - f(\eta)}{z - \eta} \right]^{\frac{1}{b}}$$

analytic and subordinate to

$$P(z) = \frac{1 + kz}{1 + z}, \dots z \in D$$

or equivalently that

$$(1.5) \quad \operatorname{Re} F(k, b, z, \eta) \geq \frac{1+k}{2}, \left| \frac{1+k}{F(k, b, z, \eta)} - 1 \right| < 1$$

Definition:

Let $f(z)$ satisfies the inequality

$$\left| \frac{f(z) - f(\eta)}{z - \eta} \right| > m, m > 0, z \in D, \eta \in D$$

then $f(z)$ is called regularly in D [2].

Coefficient Inequality For Convex Function

In this section we shall give a coefficient inequality for convex function under the regularly univalent condition.

Now we consider the inequality (This inequality is obtained from the (1.5) for $k=0, b=1$)

$$(2.1) \quad \operatorname{Re} F(0, 1, z, \eta) = \operatorname{Re} \left[\frac{f(z) - f(\eta)}{z - \eta} \right] > \frac{1}{2}$$

on the other hand, the function

$$F(0, 1, z, \eta)$$

is analytic and continuous in D; therefore, we have

$$(2.2) \quad \begin{aligned} \lim_{z \leftarrow \eta} \operatorname{Re}(F(0,1,z,\eta)) &= \lim_{z \rightarrow \eta} \left[\operatorname{Re} \frac{f(z) - f(\eta)}{z - \eta} \right] \\ &= \operatorname{Re} \left(\lim_{z \leftarrow \eta} \frac{f(z) - f(\eta)}{z - \eta} \right) = \operatorname{Re}(f'(z)) > \frac{1}{2} \end{aligned}$$

$$(2.3) \quad P(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$$

is analytic in D and satisfies $P(0) = 1, \operatorname{Re} P(z) > 0$ then $|p_n| \leq 2$. These functions are called Caratheodory functions. Considering the relations (2.2) and (2.3) together, we get

$$(2.4) \quad P(z) = 2.f(z) - 1$$

from the relation (2.4) we have

$$(2.5) \quad 2.n.a_n = p_n$$

if we use Caratheodory inequality $|p_n| \leq 2$ in the equality (2.5), we obtain

$$(2.6) \quad |a_n| \leq \frac{1}{n}$$

The inequality (2.6) is a new inequality for convex functions under the regularly univalent condition. This inequality is sharp because the function

$$f_*(z) = \operatorname{Log} \frac{1}{z-1} = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \dots + \frac{1}{n}z^n + \dots$$

is an extremal function and this function satisfies

$$\left| \frac{f_*(z) - f_*(\xi)}{z - z.\xi} \right| = \left| \frac{\operatorname{Log} \frac{1 - z.\xi}{1 - z}}{z - z.\xi} \right| \neq 0, \quad |z| < 1, |\xi| < 1, z \neq \xi$$

Therefore, the condition of regularly univalent is satisfied by this function.

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Optimization Scheme of Offshore Steel Structures

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Abstract:

The finite elements method, Reyleigh correlation and Lagrange multipliers method for the problem of optimization of offshore platform. This problem is calculated by stability, dynamic stiffness and displacement requirements.

Özet: Deniz petrol yapıların optimizasyonu için son elemanlar üsulu, Reileigh üsulu ve Lagrange katsayılar üsulları kullanılıp. Bu problem yerdeğişme, dinamik Rijitlik ve stabilite sınırı şartlı problemleri çözümlü.

Keywords: dynamic, optimization, stiffness, displacement, offshore.

Introduction

The deep-water offshore platforms of continental shelf are tremendous engineering structures. The height of this platforms reaches to and higher. The weight of structures about 400 000 kN. Therefore, the optimum design of this structure with minimum weight is an actual problem. The block of offshore platform is a space frame construction and is placed under dynamic forces: wind, waves, earthquakes, equipment installed [1]. The structures and design scheme of this platform is in fig.1. In this case the period of natural vibration of the structures becomes co-measurable with the period of external loads. Resonance occurrence is possible. Therefore, dynamic research of this structures is very necessary. Moreover, the block of offshore platform to carry upper structure with drilling oil-derrick, technological equipment and elements of structures are subjected by longitudinal bend and axis force. The structures may lose general stability.

The problem of optimum design is calculated by Lagrange multipliers method, Reileigh correlation [2].

The problem is: minimum weight

$$W = \sum_{i=1}^n \rho A_i L_i \Rightarrow \min \quad (1)$$

where: ρ -density; A_i – cross-section of i-elements; L_i –length of i-elements.

This problem is calculated by stability, dynamic stiffness and displacement requirements. The way of doing this optimisation problem is very popular and widely used in optimal design of structure [2,3,4,5]

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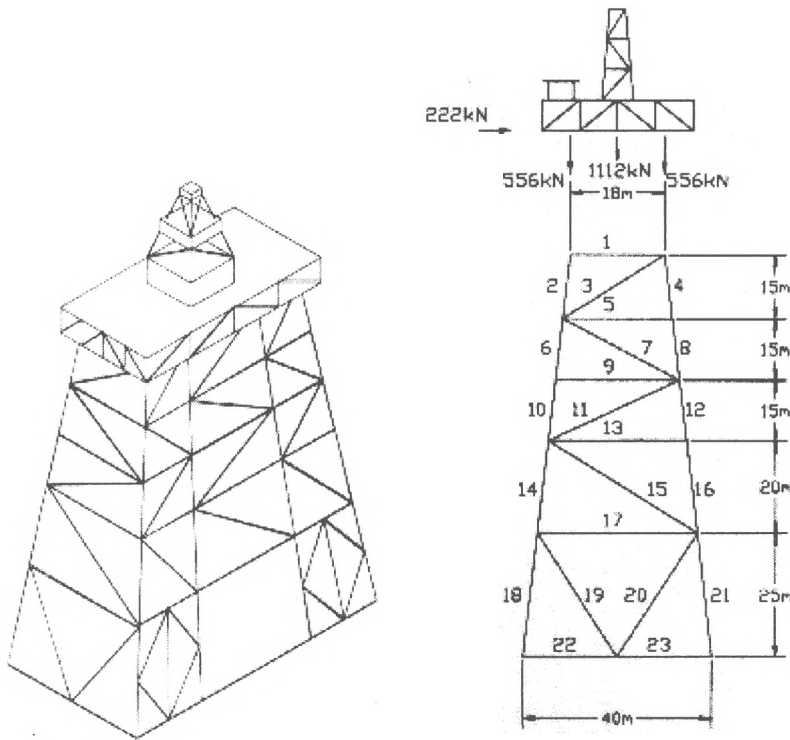


Fig.1 Structure of offshore platform (a) and design scheme (b)

Displacement Requirements

The deep-water offshore platforms are placed under horizontal forces: wind, waves, earthquake, flow, ice e.a. Therefore, displacement requirements are very actually.

The principal requirements are written in form

$$G_i = C_i - \bar{C}_i \tag{2.1}$$

In this

$$C_i = \{U\}'_i [K]_i \{S^j\}_i \tag{2.2}$$

where: $\{U\}'_i$ -displacement vector with i-element and force; $[K]_i$ -stiffness matrix of i-element.

$$\begin{bmatrix} f_{1x} \\ f_{1y} \\ m_1 \\ f_{2x} \\ f_{2y} \\ m_2 \end{bmatrix} = \frac{EA^2}{L} \begin{bmatrix} \frac{1}{A} & & & & & \\ & \textit{simmetrik} & & & & \\ 0 & \frac{12}{L^2} & & & & \\ 0 & -\frac{6}{L} & 4 & & & \\ -\frac{1}{A} & 0 & 0 & \frac{1}{A} & & \\ 0 & -\frac{12}{L^2} & \frac{6}{L} & 0 & \frac{12}{L^2} & \\ 0 & -\frac{6}{L} & 2 & 0 & \frac{6}{L} & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix} \quad (2.4)$$

$\{S^j\}_i$ -possible displacement matrix is composed by following principle:

$$K = \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix} \quad (2.5)$$

where K_{ab} reaction in a-joint from displacement b-joint.

Analytical expression for the optimum cross-section of every element is determined using the term of Lagrangian's maximum.

$$L(A, \lambda) = \sum_{i=1}^n \rho A_i L_i + \sum_{j=1}^m \lambda_j (C_j - \bar{C}_j) \quad (2.6)$$

where λ_j -Lagrange multipliers.

To get numerical solution of problem, calculation algorithm and computer program were developed.

Stability Requirements

The stability requirements are written down in form

$$G_j = \mu_j - \alpha \bar{\mu} > 0 \quad (3.1)$$

where μ_j -factic critical force with j-natural mode; $\bar{\mu}$ -lesser critical force; α -coefficient of separate mode.

$$\mu_j = \frac{\eta_j^i K \eta_j}{\eta_j^i K_g \eta_j} \quad (3.2)$$

where η_j – natural vector with j -natural mode; K_g – matrix of geometrical stiffness.

The matrix of geometrical stiffness K_g depends on internal forces by external force P from stiffness.

$$K_g = \frac{\mu}{L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ & & \frac{2L^2}{15} & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ & & & 0 & 0 & 0 \\ & & & & \frac{6}{5} & -\frac{L}{10} \\ & & & & & \frac{2L^2}{15} \end{bmatrix} \quad (3.3)$$

Dynamics Stiffness Requirements

The dynamics stiffness of structures is calculated by natural frequency. The requirements of frequency is written in form:

$$g = \omega_j^2 - \bar{\omega}^2 \quad (4.1)$$

where $\bar{\omega}$ - minimum frequency of structures.

The value of natural frequency is calculated by Reileigh method

$$\omega_j^2 = \frac{\{\psi\}' [K] \{\psi\}}{\{\psi\}' [M] \{\psi\}} \quad (4.2)$$

where $\{\psi\}$ - natural vibration mode of structure; $[K]$ - matrix of stiffness system; $[M]$ - matrix of mass with added water mass. The matrix of mass with added water mass is calculated by Reileigh discrete variation method [6].

The analysis of different calculation algorithm of optimization of structures – SAMSEF, PROSSS, TRUSSORT, SPAR, ACCESS end etc. In their study, C.Fleury, J. Sobieszczanski-Sobieski, E.Haug, L.Schmit [2,3,4,5] may come to a conclusion that the principles of all programs are finite elements method including following iteration steps:

- fixing the step for modification of cross-section area;
- composition of stiffness matrix (geometrical stiffness and matrix mass with added water mass);
- white down requirements;
- white down displacement (vibration mode, stability mode);
- white down Lagrangian for cross-section A_{j+1} and as compored with Lagrangian for cross-section A_j ;
- fixing optimum cross-section which is corresponding maximum Lagrangian.

Conclusion

The calculated result of optimum section and weight of elements by displacement, stability and dynamics stiffness are requirements on table.1.

Table 1

N elem.	L m	A m ²	W kN	Displacement		Stability		Dynamic	
				A ₀ ,m ²	W ₀ ,kN	A ₀ ,m ²	W ₀ ,kN	A ₀ ,m ²	W ₀ ,kN
1	18	0.645	905.6	0.645	905.6	0.02	28.08	0.645	905.6
2	14.9	1.138	1322.5	0.75	871.6	1.25	1452.7	0.4	464.9
3	27	0.392	825.5	0.2	421.2	0.28	589.7	0.2	421.2
4	14.9	1.138	1322.5	0.75	871.6	1.25	1452.7	0.4	464.9
5	21	0.645	1056.5	0.645	1056.5	0.02	32.76	0.645	749.6
6	14.9	1.138	1322.5	0.85	981.2	1.25	1452.7	0.51	592.7
7	30.8	0.392	941.7	0.25	600.6	0.28	672.6	0.25	600.6
8	14.9	1.138	1322.5	0.85	981.2	1.25	1452.7	0.51	592.7
9	25	0.645	1257.7	0.645	1257.8	0.02	23.2	0.645	1257.7
10	14.9	1.138	1322.5	0.95	1104.1	1.5	1743.3	0.65	755
11	34.4	0.392	1051.8	0.3	805.0	0.3	804.9	0.35	939
12	14.9	1.138	1322.5	0.95	1104.1	1.5	1743.3	0.65	755
13	29	0.645	1459	0.645	1460	0.02	45.24	0.645	1459
14	19.9	1.138	1766.4	1.05	1630	1.8	2794	0.65	1009
15	37.3	0.392	1140.5	0.35	1018.3	0.28	814.6	0.35	1018
16	19.9	1.138	1766.4	1.05	1630	1.8	2794	0.65	1009
17	34	0.645	1710.5	0.645	1710.5	0.02	53.04	0.645	1710
18	24.8	1.138	2201.3	1.15	2224.6	2.2	4255.6	0.65	1257
19	30.2	0.392	923.4	0.35	824.5	0.28	659.5	0.35	824.4
20	30.2	0.392	923.4	0.35	824.5	0.28	659.5	0.35	824.4
21	24.8	1.138	2201.3	1.15	2224.6	2.2	4255.6	0.65	1257
22	20	0.645	1006	0.645	1006	0.22	343.2	0.645	1006.2
23	20	0.645	1006	0.645	1006	0.22	343.2	0.645	1006.2

30078

26519.5

28466.12

20879

Optimization Scheme of Offshore Steel Structures

The analysis of result may notice that in case stability requirements cross-section of horizontal elements is almost equal to zero. Then we can remove it and so change topological scheme of structures. That is its may throw aside and all topological scheme of structures change. In this variant, cross-section of vertical elements is very large and therefore, its case are no profitable.

The most profitable are case of dynamics stiffness requirement.

Economical effect with difference of weight for one panel is calculated:

Displacement

Requirements
$$W = \frac{W_0 - W}{W_0} 100\% = \frac{30078 - 26519.6}{30078} 100\% = 12\%$$

Stability

Requirements
$$W = \frac{W_0 - W}{W_0} 100\% = \frac{30078 - 28466.12}{30078} 100\% = 5.3\%$$

Dynamics

Stiffness
$$W = \frac{W_0 - W}{W_0} 100\% = \frac{30078 - 20879}{30078} 100\% = 30.6\%$$

requirements.

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A New Method for Increasing the Earthquake Safety of the Structures Seismic Isolation

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Abstract

Seismic isolation provides an excellent solution for the earthquake safety of the structures. This paper explains basic principles of this approach.

Özet

Sismik izolasyon, yapıların deprem emniyetinin sağlanması için mükemmel bir çözümdür. Bu makale, sismik izolasyon yaklaşımının temel ilkelerini açıklamaktadır.

Keywords

Introduction

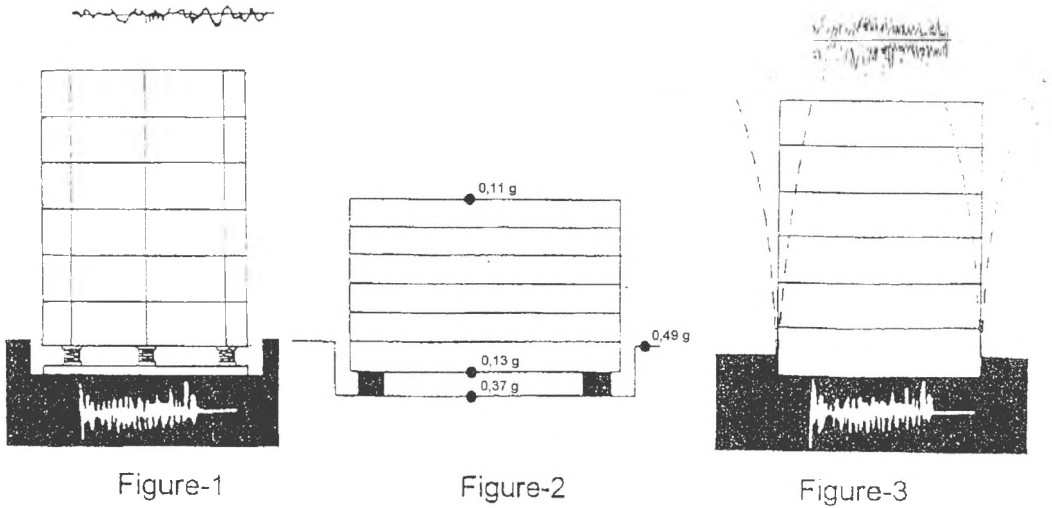
An earthquake is a natural phenomenon. They are bound to occur at various intervals. The important issue here is to mitigate the loss of lives and possessions by building earthquake resistant structures or by decreasing the effects of earthquake induced forces. Seismic isolation technique has been developed greatly in the last 25-30 years. During the 1994 Northridge (USA) and 1995 Kobe (Japan) earthquakes, the buildings with seismic isolation performed extremely well under the earthquake induced forces[1]. These buildings did not collapse and expensive equipment inside was not harmed. These results clearly increased the popularity of the seismic isolation technique. Led by the USA, many nations now require that hospitals, facilities of the fire brigade and telecommunication and bridges that must continue to serve after an earthquake be protected by seismic isolation. In Turkey, the new earthquake code that was revised in 1998 states that seismic isolation can be applied on structures and it is adequate to conform to the American Earthquake Code[3].

Seismic Isolation Method

Seismic isolation technique is based on a simple principle. The seismic isolators placed between the foundation and the vertical load carrying elements, namely columns and where available shear walls, allow the supports to move in the horizontal direction (Figure-1). Therefore by means of seismic isolation, the earthquake acceleration induced on the structure from the ground is decreased up to 80 percent and both the structure itself and the equipment inside is left unharmed

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(Figure-2).



But without the seismic isolators, the ground accelerations would continue to increase on the upper floors and cause significant horizontal displacements and damage on the structure (Figure-3).

Seismic isolation technique has completed the stages of scientific and technical research and found its rightful place in the earthquake codes. The process is completed by placing the isolators, which are now produced in many countries, during the construction period in accordance with the structural designs [2]. Although it is easier and more economical to place the isolators during the construction period, it is also possible to place them to existing structures to increase their seismic safety.

In Turkey, the seismic isolation technique was first applied at the new terminal building of the Ataturk Airport in Istanbul to provide additional seismic safety after the 17 August 1999 earthquake. However, the seismic isolators were placed at the top of columns beneath the supports of the steel roof construction with large spans at the top floor since the construction period was already completed at the time.

Although still in the project stages, preparations are being made to use seismic isolation at some hospitals, telecommunication facilities, factories, domiciles, bridges, overpasses and liquefied natural gas tanks.

Among the structures that must be fortified with seismic isolation, liquefied natural gas tanks are one of the most significant types of structures from the environmental point of view. In Greece, LNG tanks with diameter of 70 meters and a height of 32 meters were provided with seismic isolation to increase their seismic safety.

An Application from Turkey

Starting with the ones in Marmara Ereglisi, liquefied natural gas tanks at Marmara and Aegean regions must be provided with the seismic isolation to increase their existing seismic safety.

It is natural for the seismic isolation to increase the production cost of a structure. However, superstructure is significantly economized since the earthquake forces are decreased up to 80 percent. As an example, the load carrying components of the ongoing construction of the new terminal building of the San Francisco Airport are made of steel. 600 tons of steel on the superstructure alone was saved thanks to seismic isolation. In this case, the cost of seismic isolation is decreased at a significant rate.

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Weyl Orbit Characters and Schur Functions

H. R. Karadayı* and M. Güngörmez**

Abstract

For finite Lie algebras, it is shown that characters can be defined first for Weyl orbits and then for irreducible representations. For A_N Lie algebras, weight multiplicities can then be calculated by only stating that characters are equivalent to Schur functions. This also means that to calculate characters there is no need to sum over Weyl groups. The compatibility with the Weyl character formula will however be shown.

Özet

Sonlu Lie cebirleri karakterlerinin ilk olarak Weyl yörüngeleri ve daha sonra indirgenemez temsilleri için tanımlanabildiği gösterilmektedir. Böylelikle, A_N Lie cebirleri için, ağırlık çokkathlıkları karakterlerin doğrudan Schur fonksiyonlarına eşdeğer olduğu öngörüsü yardımıyla hesaplanabilmektedir. Bunun bir anlamı da, karakterleri hesaplamak için Weyl grubu üzerinden toplama ihtiyacı olmadığıdır. Bunun Weyl karakter formülü ile uyumluluğu ayrıca gösterilmektedir.

Keywords: Weyl Orbits, Schur Functions.

I. Introduction

We begin by considering Schur functions which are defined [1] by

$$\sum_{M=0}^{\infty} S_M(x_1, x_2, \dots, x_M) z^M \equiv \text{Exp} \sum_{i=1}^{\infty} x_i z^i \quad (I.1)$$

where $S_0=1$. This definition includes only some restricted class of Schur functions $S_M(x_1, x_2, \dots, x_M)$ which are polynomials of M indeterminates x_1, x_2, \dots, x_M and also homogeneous of degree M . Let,

$$(q_1, q_2, \dots, q_N)$$

be a partition into N parts. It is called a partition with weight M providing

$$q_1 + q_2 + \dots + q_N \equiv M \quad , \quad q_1 \geq q_2 \geq \dots \geq q_N \quad (I.2)$$

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Note that there could always be cases for which some of integers q_1, q_2, \dots, q_N are chosen to be zero. For any such partition, one can make a generalization

$$S_{(q_1, q_2, \dots, q_N)}(x_1, x_2, \dots, x_M) \equiv \text{Det} \begin{pmatrix} S_{q_1-0} & S_{q_1+1} & S_{q_1+2} & \dots & S_{q_1+M-0} \\ S_{q_2-1} & S_{q_1+0} & S_{q_2+1} & \dots & S_{q_2+M-1} \\ S_{q_3-2} & S_{q_3-1} & S_{q_3+0} & \dots & S_{q_3+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad (I.3)$$

where Det means determinant. Each and every matrix element S_k is assumed here to be a function of only k independent parameters x_1, x_2, \dots, x_k where $k \leq M$ and $S_{k < 0} = 0$. Note here that both $S_M(x_1, x_2, \dots, x_M)$ and also $S_{(q_1, q_2, \dots, q_N)}(x_1, x_2, \dots, x_M)$ are called Schur functions.

This is, in fact, not the whole story. Some degeneration occurs in the definition of Schur functions when one wants to identify them with the characters of irreducible representations of A_N Lie algebras. After introducing the Weyl orbit characters in the next section, the degeneration of Schur functions will be briefly explained in section III and the formula

(III.7) will be given. In section IV, we show how one calculates weight multiplicities by identifying characters directly with Schur functions. It will be seen that, beside the known ones [2], this gives us a multiplicity formula for A_N Lie algebras. The compatibility of all these with the Weyl character formula will be given in the last section.

II. Characters For A_{N-1} Weyl Orbits

Characters are conventionally known to be defined for irreducible representations. Now we will show that they can be defined first for Weyl orbits and then for representations. Here, it is essential to use **fundamental weights** μ_i ($i=1, 2, \dots, N$) which are defined, for A_{N-1} Lie algebras, by

$$\begin{aligned} \mu_1 &\equiv \lambda_1 \\ \mu_i &\equiv \mu_{i-1} - \alpha_{i-1} \quad , \quad i=2, 3, \dots, N \end{aligned} \quad (II.1)$$

or conversely by

$$\lambda_i = \mu_1 + \mu_2 + \dots + \mu_i \quad , \quad i=1, 2, \dots, N-1. \quad (II.2)$$

Weyl Orbit Characters and Schur Functions

together with the condition that

$$\mu_1 + \mu_2 + \dots + \mu_N \equiv 0 \tag{II.3}$$

λ_i 's and α_i 's ($i=1,2,\dots,N-1$) are **fundamental dominant weights** and **simple roots** of

A_{N-1} Lie algebras. For an excellent study of Lie algebra technology we refer to the book of Humphreys [3]. We know that there is an irreducible A_{N-1} representation for each and every dominant weight Λ^+ which can be expressed by

$$\Lambda^+ = \sum_{i=1}^N q_i \mu_i \quad , \quad q_1 \geq q_2 \geq \dots \geq q_N \geq 0 \tag{II.4}$$

One thus concludes that there is a dominant weight Λ^+ for each and every partition (q_1, q_2, \dots, q_N) with weight $M = q_1 + q_2 + \dots + q_N$. The following definition will then be useful.

Definition:

Let N, M and k be positive integers such that $M - kN \leq M$. Then, we state $\text{Sub}(M \lambda_1)$ to be the set of dominant weights corresponding, via (II.4), to all partitions of $M - kN$ for $k=0,1,2, \dots$

It is now clear that $\text{Sub}(M \lambda_1)$ is consist of $M \lambda_1$ and all of its sub-dominant weights. The weight structure of the corresponding irreducible representation $R(\Lambda^+)$ can then be expressed by the aid of the following **orbital decomposition**:

$$R(\Lambda^+) = \sum_{\lambda^+ \in \text{Sub}(M \lambda_1)} m(\lambda^+ \prec \Lambda^+) W(\lambda^+) \tag{II.5}$$

where $m(\lambda^+ \prec \Lambda^+)$ is the multiplicity of Λ^+ within $R(\Lambda^+)$. Note here that one always has

$m(\lambda^+ \prec \Lambda^+) = 1$ and also

$$m(\lambda^+ \prec \Lambda^+) = 0 \tag{II.6}$$

if λ^+ is not a sub-dominant weight of Λ^+ . For instance, the tensor representations with M completely symmetric and antisymmetric indices correspond, respectively, to two extreme cases which come from the partitions of M into 1 and M non-zero parts provided that

$$m(\lambda^+ \prec M \Lambda^+) = 1 \quad (\text{II.7})$$

and

$$m(\lambda^+ \prec \lambda_M) = 0 \quad , \quad \lambda^+ \neq \lambda_M \quad (\text{II.8})$$

for $\lambda^+ \in \text{Sub}(M \lambda_1)$. They are, in fact, nothing but the representations $R(M \lambda_1)$ and $R(\lambda_M)$, respectively. The characters can now be expressed by

$$\text{Ch}R(\Lambda^+) = \sum_{\lambda^+ \in \text{Sub}(M \lambda_1)} m(\lambda^+ \prec \Lambda^+) \text{Ch}W(\lambda^+) \quad (\text{II.9})$$

In view of orbital decomposition (II.5), it is clear that (II.9) allows us to define the characters

$$\text{Ch}W(\Lambda^+) = \sum_{\mu \in W(\Lambda^+)} e(\mu) \quad (\text{II.10})$$

for Weyl orbits $W(\Lambda^+)$. **Formal exponentials** $e(\mu)$ are defined, for any weight μ , as in the book of Kac [4]. We refer here to a lemma which we previously introduced [5].

Lemma:

Let Λ^+ be an A_{N-1} dominant weight which we know to be identified with a partition (q_1, q_2, \dots, q_N) . Then, any other weight $\omega(\Lambda^+) \in W(\Lambda^+)$ is obtained by permutating the parameters q_1, q_2, \dots, q_N , that is

$$\omega(\Lambda^+) = q_{i_1} \mu_1 + q_{i_2} \mu_2 + \dots + q_{i_N} \mu_N$$

where $i_1, i_2, \dots, i_N = 1, 2, \dots, N$ and the Weyl orbit $W(\Lambda^+)$ is formed out of all possible permutations.

One could say that such a lemma is trivial in view of the fact that Weyl groups of A_{N-1} algebras are nothing but the permutation groups. One must note however that this can be seen only by using the so-called fundamental weights defined above.

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We now introduce the generators defined by

$$K_M(u_1, u_2, \dots, u_N) \equiv u_1^M + u_2^M + \dots + u_N^M \quad (\text{II.11})$$

and their generalizations

$$K_{(q_1, q_2, \dots, q_N)}(u_1, u_2, \dots, u_N) \equiv \sum_{j_1, j_2, \dots, j_N=1}^N (u_{j_1})^{q_1} (u_{j_2})^{q_2} \dots (u_{j_N})^{q_N} \quad (\text{II.12})$$

for any partition (q_1, q_2, \dots, q_N) with weight M . In (II.12), no any two of indices j_1, j_2, \dots, j_N shall take the same value for each particular monomial. For some other reasons, we would like to call them **class functions**. On the contrary to generic definition of Schur Functions, the values of M and N need not be correlated in both of these expressions. One must note here that, as a result of its very definition given in (II.12), any class function $K_{(q_1, q_2, \dots, q_N)}$ has always a non-linear expansion in terms of generators K_M defined in (II.11). We call these expansions **reduction rules**. One can give

$$K_{(q_1, q_2)} = K_{q_1} K_{q_2} - K_{q_1+q_2} \quad , \quad q_1 > q_2 \quad (\text{II.13})$$

$$K_{(q_1, q_2)} = \frac{1}{2} K_{q_1} K_{q_2} - \frac{1}{2} K_{q_1+q_2}$$

as simple but instructive examples of reduction rules though more advanced ones can be obtained in exactly the same manner.

It is now seen that (II.10) is equivalent to

$$\text{ChW}(\Lambda^+) = K_{(q_1, q_2, \dots, q_N)}(u_1, u_2, \dots, u_N) \quad (\text{II.14})$$

where we choose the specialization

$$e(\mu_i) \equiv u_i \quad , \quad i = 1, 2, \dots, N. \quad (\text{II.15})$$

Due to (II.3), the indeterminates u_1, u_2, \dots, u_N here are constrained by the condition that

$$u_1 u_2 \dots u_N \equiv 1 \quad (\text{II.16})$$

which can be expressed in the following equivalent form:

$$K_{(1, 1, \dots, 1)}(u_1, u_2, \dots, u_N) \equiv 1 \quad (\text{II.17})$$

III. Degenerated Schur Functions

Now we are know that, for A_{N-1} , there could only be N-1 generic Schur functions

$$S_1(x_1), S_2(x_1, x_2), \dots, S_{N-1}(x_1, x_2, \dots, x_{N-1})$$

which can be identified from definition (I.1). For $M \geq N$, on the other hand, the Schur functions

$$S_M(x_1, x_2, \dots, x_{N-1}, x_N, \dots, x_M)$$

transform into some degenerated ones

$$S_{M \geq N}(x_1, x_2, \dots, x_{N-1}) \tag{III.1}$$

due to the fact that one can only have N-1 indeterminates. For this, one must first find an appropriate equivalence between the sets x_1, x_2, \dots, x_{N-1} and u_1, u_2, \dots, u_N of indeterminates. Hint comes from completely symmetric functions $h_k(u_1, u_2, \dots, u_N)$ defined by

$$\sum_{k=0}^{\infty} h_k(u_1, u_2, \dots, u_N) z^k \equiv \prod_{i=1}^N \frac{1}{(1 - z u_i)} \tag{III.2}$$

They can be given equivalently by

$$h_M(u_1, u_2, \dots, u_N) = \sum K_{(q_1, q_2, \dots, q_N)}(u_1, u_2, \dots, u_N) \tag{III.3}$$

where the sum is over all partitions (q_1, q_2, \dots, q_N) with weight M. The main observation here is that

$$h_M(u_1, u_2, \dots, u_N) \equiv S_M(x_1, x_2, \dots, x_M) \tag{III.4}$$

is obtained [1] by the aid of replacements

$$K_M(u_1, u_2, \dots, u_N) \equiv M x_M \tag{III.5}$$

From the now on, two sets of indeterminates will always be thought of as the same via correspondences (III.5). The condition (II.17) can be extended, on the other hand, as in the following:

$$K_{(Q, 1, \dots, 1)}(u_1, u_2, \dots, u_N) \equiv K_Q(u_1, u_2, \dots, u_N) \quad , \quad Q=1, 2, \dots \tag{III.6}$$

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In view of the reduction rules mentioned above, this gives us a way to introduce extra indeterminates $x_{(Q+N-1)}$ which are non-linear polynomial solutions of (III.6) in terms of indeterminates x_1, x_2, \dots, x_N . We have found that the result of all these calculations leads us to the following formula for the degenerated Schur functions specified as in (III.1):

$$S_M = (-1)^N S_{M-N-1} - \sum_{k=1}^N S_k^* S_{M-k} \quad , \quad M \geq N \quad (\text{III.7})$$

where S_k^* is obtained from S_k under replacements $x_i \rightarrow -x_i$ ($i=1,2,\dots,N-1$). Let us note here that this formula has a central importance for following applications of our method.

IV. Explicit Calculation of Weight Multiplicities

Let

$$\Lambda^+ \equiv k_1 \mu_1 + k_2 \mu_2 + \dots + k_N \mu_N \in \text{Sub}(M \lambda_1) \quad (\text{IV.1})$$

be a dominant weight. In view of (II.14), its character (II.9) will now be given by

$$\text{Ch}(\Lambda^+) = \sum_{m_{(q_1, q_2, \dots, q_N)}} (\Lambda^+) K_{(q_1, q_2, \dots, q_N)}(u_1, u_2, \dots, u_N) \quad (\text{IV.2})$$

where $m(\lambda^+ \prec \Lambda^+) \equiv m_{(q_1, q_2, \dots, q_N)}(\Lambda^+)$

is assumed for convenience in the notation. The sum in (IV.2) is over all permutations (q_1, q_2, \dots, q_N) with weight M for which

$$q_1 \mu_1 + q_2 \mu_2 + \dots + q_N \mu_N \in \text{Sub}(M \lambda_1).$$

We also know that (IV.1) gives us a Schur function

$$S_{(k_1, k_2, \dots, k_N)}(x_1, x_2, \dots, x_{N-1}).$$

Now by stating that

$$\text{Ch}(\Lambda^+) = S_{(k_1, k_2, \dots, k_N)}(x_1, x_2, \dots, x_{N-1}) \quad (\text{IV.3})$$

one obtains, from the equality of (IV.2) and (IV.3), the **multiplicity formula**

$$\sum_{m_{(q_1, q_2, \dots, q_N)}} (\Lambda^+) K_{(q_1, q_2, \dots, q_N)}(u_1, u_2, \dots, u_N) = S_{(k_1, k_2, \dots, k_N)}(x_1, x_2, \dots, x_{N-1}). \quad (\text{IV.4})$$

By using the reduction rules mentioned above, this formula turns to an equation for monomials

$$x_1^{j_1} x_2^{j_2} \dots x_{N-1}^{j_{N-1}}$$

where $j_1, j_2, \dots, j_{N-1} = 0, 1, 2, \dots$ are constrained by

$$j_1 + j_2 + \dots + j_{N-1} = M - kN \geq 0, \quad k=0, 1, 2, \dots \quad (\text{IV.5})$$

due to homogeneity. It will depend linearly on multiplicities and the number of these multiplicities will be equal to the number of monomials given above. This hence gives us, for any choice of $\Lambda^+ \in \text{Sub}(M \lambda_1)$, the possibility to compute all the multiplicities by solving a system of linear equation.

Let us visualize our framework, for A_4 Lie algebra, in the moderated example of $\text{Sub}(6 \lambda_1)$ which consists of the following ten dominant weights:

$$\begin{aligned} (6, 0, 0, 0, 0) &\rightarrow 6 \lambda_1 \\ (5, 1, 0, 0, 0) &\rightarrow 4 \lambda_1 + \lambda_2 \\ (4, 2, 0, 0, 0) &\rightarrow 2 \lambda_1 + 2 \lambda_2 \\ (3, 3, 0, 0, 0) &\rightarrow 3 \lambda_2 \\ (4, 1, 1, 0, 0) &\rightarrow 3 \lambda_1 + \lambda_3 \\ (3, 2, 1, 0, 0) &\rightarrow \lambda_1 + \lambda_2 + \lambda_3 \\ (2, 2, 2, 0, 0) &\rightarrow 2 \lambda_3 \\ (3, 1, 1, 1, 0) &\rightarrow 2 \lambda_1 + \lambda_4 \\ (2, 2, 1, 1, 0) &\rightarrow 2 \lambda_1 + \lambda_4 \\ (1, 0, 0, 0, 0) &\rightarrow \lambda_1 \end{aligned} \quad (\text{IV.6})$$

where, due to definitions (II.1) or (II.2), corresponding partitions are given on the left-hand side. For A_4 , we have only four Schur functions

$$S_1(x_1), S_2(x_1, x_2), S_3(x_1, x_2, x_3), S_4(x_1, x_2, x_3, x_4).$$

Any other $S_M(x_1, x_2, x_3, x_4)$ will however be a degenerated Schur function for $M \geq 5$. In this example, we need only $S_5(x_1, x_2, x_3, x_4)$ and $S_6(x_1, x_2, x_3, x_4)$. From definition (I.1), it is known that

$$S_5(x_1, x_2, x_3, x_4, x_5) \equiv \frac{1}{120} x_1^5 + \frac{1}{6} x_1^3 x_2 + \frac{1}{2} x_1 x_2^2 + \frac{1}{2} x_1^2 x_3 + x_2 x_3 + x_1 x_4 + x_5 \quad (\text{IV.7})$$

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On the other hand, by solving (II.17), one obtains

$$x_5 = 1 - \frac{1}{120}x_1^5 + \frac{1}{6}x_1^3x_2 - \frac{1}{2}x_1x_2^2 - \frac{1}{2}x_1^2x_3 + x_2x_3 + x_1x_4 \quad (\text{IV.8})$$

(IV.7) and (IV.8) now give us the degenerated Schur function

$$S_5(x_1, x_2, x_3, x_4) = 1 + \frac{1}{3}x_1^3x_2 + 2x_2x_3 + 2x_1x_4. \quad (\text{IV.9})$$

In the same manner, one also obtains

$$S_6(x_1, x_2, x_3, x_4) = 2x_1 - \frac{1}{72}x_1^6 + \frac{1}{3}x_1^4x_2 - \frac{1}{2}x_1^2x_2^2 - \frac{2}{3}x_1^3x_3 + 2x_1x_2x_3 + x_3^2 + 2x_1^2x_4 + 2x_1x_4 \quad (\text{IV.10})$$

from

$$K_{(2,1,1,1,1)}(u_1, u_2, \dots, u_5) = K_1(u_1, u_2, \dots, u_5).$$

One easily sees that both (IV.9) and (IV.10) are compatible with (III.7). This means that, beside their very definitions given above, they can also be extracted from (III.7) directly.

It is now seen that, from (IV.4), one obtains a linear equation for the following ten monomials

$$x_1, x_1^6, x_1^4x_2, x_1^3x_3, x_1^2x_2^2, x_1^2x_4, x_2^3, x_2^2, x_2x_4, x_1x_2x_3$$

where their coefficients are linearly depend on the ten multiplicities corresponding to dominant weights given in (IV.5). Let us choose $S_{(6,0,0,0,0)}(x_1, x_2, x_3, x_4)$ on the right hand side of (IV.4). The linear independence of above monomials then gives rise to a result which is compatible with (II.7). As another example, let us choose

$S_{(5,1,0,0,0)}(x_1, x_2, x_3, x_4)$ on the right hand side of (IV.4). Then one has following solutions for corresponding multiplicities:

$$\begin{aligned} m(6\lambda_1 < 4\lambda_1 + \lambda_2) &= 0, \\ m(4\lambda_1 + \lambda_2 < 4\lambda_1 + \lambda_2) &= 1, \\ m(2\lambda_1 + 2\lambda_2 < 4\lambda_1 + \lambda_2) &= 1, \\ m(3\lambda_2 < 4\lambda_1 + \lambda_2) &= 1, \\ m(3\lambda_1 + \lambda_3 < 4\lambda_1 + \lambda_2) &= 2, \\ m(\lambda_1 + \lambda_2 + \lambda_3 < 4\lambda_1 + \lambda_2) &= 2, \\ m(2\lambda_3 < 4\lambda_1 + \lambda_2) &= 2, \\ m(2\lambda_1 + \lambda_4 < 4\lambda_1 + \lambda_2) &= 3, \\ m(\lambda_2 + \lambda_4 < 4\lambda_1 + \lambda_2) &= 3, \\ m(\lambda_1 < 4\lambda_1 + \lambda_2) &= 4. \end{aligned}$$

V. Weyl Character Formula

Weyl character formula simply says that

$$ChR(\Lambda^+) = \frac{A(\rho + \Lambda^+)}{A(\rho)} \quad (V.1)$$

where $\rho \equiv \lambda_1 + \lambda_2 + \dots + \lambda_{N-1}$ is the **Weyl vector** of A_{N-1} . If one assumes that the left-hand side of (V.1) is determined by (II.9), then we will show that its right-hand side gives rise just to the same result hence the formula. The main object is

$$A(\Lambda^+) \equiv \sum_{\omega} \varepsilon(\omega) e^{\omega(\Lambda^+)} \quad (V.2)$$

where the sum is over A_{N-1} Weyl group and hence $\omega(\Lambda^+)$ represents an action ω of the Weyl group. Instead, we will consider (V.2) always in the following equivalent form:

$$A(\Lambda^+) \equiv \sum_{\mu \in W(\Lambda^+)} \varepsilon(\mu) e^{\mu} . \quad (V.3)$$

In (V.2), $\varepsilon(\omega)$ is called the **signature** which is known to be defined by

$$\varepsilon(\omega) = (-1)^{\ell(\omega)} \quad (V.4)$$

where $\ell(\omega)$ is the minimum number of simple reflections [3] to obtain a Weyl reflection ω . We will however give another definition of the signature and it is seen that this could be more convenient in explicit calculations. Having in mind the lemma given in section II, let us now consider the dominant weight Λ^+ in the following form

$$\Lambda^+ = q_1 \mu_1 + q_2 \mu_2 + \dots + q_s \mu_s \quad (V.5)$$

For $s \leq N$, it is seen that (V.5) is completely equivalent to (II.4). Having in mind the lemma given in section II, an element μ of its Weyl orbit can be expressed as in

$$\mu = q_{i_1} \mu_1 + q_{i_2} \mu_2 + \dots + q_{i_s} \mu_s \quad (V.6)$$

where

$$q_1 \geq q_2 \geq \dots \geq q_s > 0$$

We can thus replace (V.4) by

$$\varepsilon(\mu) = \varepsilon_{q_{i_1}, q_{i_2}, \dots, q_{i_s}} \quad (V.7)$$

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The tensor $\varepsilon_{q_1, q_2, \dots, q_s}$ is completely antisymmetric in its indices while its numerical value is given by the condition that

$$\varepsilon(\Lambda^+) \equiv \varepsilon_{q_1, q_2, \dots, q_s} = +1$$

In the specialization (II.15) given above, it is now seen that

$$A(\rho) = \prod_{j>l=1}^N (u_l - u_j) \quad (\text{V.8})$$

which is in fact nothing but the Vandermonde determinant. It is also known that $A(\rho + \Lambda^+)$ is equivalent to determinant of the following $N \times N$ matrix:

$$\begin{pmatrix} u_1^{q_1-1+N} & u_1^{q_2-2+N} & \dots & u_1^{q_N-N+N} \\ u_2^{q_1-1+N} & u_2^{q_2-2+N} & \dots & u_2^{q_N-N+N} \\ \vdots & \vdots & \vdots & \vdots \\ u_N^{q_1-1+N} & u_N^{q_2-2+N} & \dots & u_N^{q_N-N+N} \end{pmatrix} \quad (\text{V.9})$$

As the main result of this section, one can show that (V.8) and (V.9) give

$$A(\rho + \Lambda^+) = A(\rho) S_{(q_1, q_2, \dots, q_N)}(x_1, x_2, \dots, x_{N-1}) \quad (\text{V.10})$$

for the dominant weight $\Lambda^+ = q_1 \mu_1 + q_2 \mu_2 + \dots + q_N \mu_N$. A nice example can be given in the case of $\Lambda^+ = M \lambda_1$ by

$$A(\rho + M \lambda_1) = A(\rho) S_M(x_1, x_2, \dots, x_{N-1}) \quad (\text{V.11})$$

which is nothing but a simple result of (II.7). Note here that, for $M \geq N$, (V.11) always gives us a degenerated schur function. For any other $\Lambda^+ \in \text{Sub}(M \lambda_1)$, (V.10) gives on the right hand side the Schur functions which can be obtained via (I.3). In result, this shows us that our statement (IV.3) is in complete agreement with Weyl character formula.

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