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STARLIKE FUNCTIONS ASSOCIATED WITH $\tanh z$ AND BERNARDI INTEGRAL OPERATOR

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Dedicated to Prof. Vijay Gupta on the occasion of his 60th birthday

Abstract. We determine the necessary and sufficient convolution conditions for the starlike functions on the open unit disk and related to some geometric aspects of the function $\tanh z$. We also determine sharp bounds on second and third order Hermitian-Toeplitz determinants for such functions. Further, we compute estimates on some initial coefficients and the Hankel determinants of third and fourth order. In addition, using the concept of Briot-Bouquet type differential subordination, we establish a subordination inclusion involving Bernardi integral operator.

1. **Introductory literature and the class $S^*_{\tanh}$.** Let $\mathcal{A}$ be the class of normalized analytic functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined in $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$. The subclass $\mathcal{S}$ of the class $\mathcal{A}$ contains all the univalent functions. If g, $h \in \mathcal{A}$, where $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ and $h(z) = z + \sum_{n=2}^{\infty} h_n z^n$, then convolution or Hadamard product $g * h$ of g and h is defined as $g(z) * h(z) = (g * h)(z) = z + \sum_{n=2}^{\infty} g_n h_n z^n$. If $h(z) = z/(1-z)$, then $g * h = g$ and if $h(z) = z/(1-z)^2$, then $g * h = zg'$ for all $f \in \mathcal{A}$ [11]. The convolution properties for various subclasses have discussed in [8,13]. For the function $f \in \mathcal{A}$, the q^{th} Hermitian-Toeplitz determinant is given by $HT_q(n) := [a_{ij}]$, where $a_{ij} = a_{n+j-i}$ for $j \geq i$ and $a_{ij} = \overline{a_{ji}}$ for $j < i$. The second and third Hermitian-Toeplitz determinant containing the initial coefficients in the series expansion of the functions $f \in \mathcal{A}$ are given by

$$HT_2(1)(f) := 1 - |a_2|^2 \quad \text{and} \quad HT_3(1)(f) := 2\Re(a_2^2\,\bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1. \quad (1)$$

The q^{th} Hankel determinant $H_q(n)(f)$ associated with the initial coefficients in the series expansion of functions $f \in \mathcal{A}$ is given as $H_q(n) := \det\{a_{n+i+j-2}\}_{i,j}^{q}$,

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$1 \leq i,\, j \leq q$, where $a_1 = 1$. Thus, the third and fourth order Hankel determinants are given by

$$H_3(1)(f) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) + a_5(a_3 - a_2^2) \tag{2}$$
$$H_3(2)(f) = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4H_2(3)(f) \tag{3}$$

$$\begin{aligned} H_4(1)(f) =& a_7H_3(1)(f) - a_6((a_3a_6 - a_4a_5) - a_2(a_2a_6 - a_3a_5) + a_4H_2(2)(f)) \\ &+ a_5((a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3H_2(3)(f)) - a_4H_3(2)(f) \\ =& a_7H_3(1)(f) - a_6\delta_1 + a_5\delta_2 - a_4H_3(2)(f), \end{aligned} \tag{4}$$

where $\delta_1 = (a_3a_6 - a_4a_5) - a_2(a_2a_6 - a_3a_5) + a_4H_2(2)(f)$ and $\delta_2 = (a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3H_2(3)(f)$. For more details related to Hermitian-Toeplitz and Hankel determinants, see [20, 37].

The analytic function f_1 is subordinate to an analytic function f_2, symbolized by $f_1 \prec f_2$, if there is an analytic function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f_1 = f_2 \circ w$. For the analytic function ϕ satisfying $\phi(0) = \varphi(0) = 1$, the Briot-Bouquet differential subordination is given by

$$\phi(z) + \frac{z\phi'(z)}{\eta\phi(z) + \mu} \prec \varphi(z), \quad (\eta, \mu \in \mathbb{C}, \eta \neq 0). \tag{5}$$

If the function $q(z) = 1 + q_1z + q_2z^2 + ...$ satisfies relation $\phi \prec q$, then q is known as a dominant of (5). If the subordination relation $\tilde{q} \prec q$ holds for all dominant q, then $\tilde{q}$ is the best dominant. In [12], the Briot-Bouquet differential subordination related results involving best dominant were discussed. For more recent best dominant related results, see [6, 7, 24]. The Bernardi integral operator [5] $\mathcal{L}_\sigma : \mathcal{A} \to \mathcal{A}$ with $\sigma > 0$ is given by

$$\mathcal{L}_\sigma f(z) = \frac{1+\sigma}{z^\sigma} \int_0^z t^{\sigma-1} f(t)dt.$$

Using this operator, we get

$$z(\mathcal{L}_\sigma f(z))' = (1+\sigma)f(z) - \sigma\mathcal{L}_\sigma f(z). \tag{6}$$

In [30], author discussed the starlikeness of order α for the Bernardi integral operator and authors [3] apply Briot-Bouquet differential subordination to this integral operator of the class of the Janowski starlike functions. Recently, Gupta [14] discussed a generalised class of integral operators. For more information related to operators, see [1, 2, 15]. The class $\mathcal{P}$ contains all analytic functions p defined on $\mathbb{D}$ satisfying conditions $p(0) = 1$ and $\Re p(z) > 0$. Many authors computed the best possible estimates on various coefficients functionals for the functions $p \in \mathcal{P}$ which are given as:

Lemma 1.1. [28, Lemma 3, p. 254] *Let*

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots \tag{7}$$

be in the class $\mathcal{P}$ *and satisfy* $\Re p(z) > 0$ $(z \in \mathbb{D})$. *Then* $2p_2 = p_1^2 + (4 - p_1^2)\xi$ *for some* $\xi \in \overline{\mathbb{D}}$.

Lemma 1.2. [33, Lemma 2.3, p. 507] *Let* $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots \in \mathcal{P}$. *Then*

$$|\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & \textit{elsewhere}, \end{cases}$$

for all $n, m \in \mathbb{N}$. *If* $0 < \mu < 1$, *then equality holds for the function* $p(z) = (1 + z^{m+n})/(1 - z^{m+n})$. *In all other cases, equality holds for the function* $p_0(z) = (1 + z)/(1 - z)$.

Lemma 1.3. [25, Lemma 2.1, p. 1055] *Let* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$. *Then*

$$|\mu p_3 - p_1^3| \leq \begin{cases} 2|\mu - 4|, & \mu \leq \frac{4}{3}; \\ 2\mu\sqrt{\frac{\mu}{\mu-1}}, & \mu > \frac{4}{3}, \end{cases}$$

for any real number μ. *The result is sharp. If* $\mu \leq \frac{4}{3}$, *equality holds for the function* $p_0(z) := (1 + z)/(1 - z)$, *and if* $\mu > \frac{4}{3}$, *then equality holds for the function*

$$p_1(z) := \frac{1 - z^2}{z^2 - 2\sqrt{\frac{\mu}{\mu-1}}\, z + 1}.$$

The function $f \in \mathcal{S}$ is starlike if $f(\mathbb{D})$ is starlike domain with respect to origin or the function $f \in \mathcal{S}$ is starlike if and only if $zf'(z)/f(z) \in \mathcal{P}$ for all $z \in \mathbb{D}$. Several subclasses of the starlike functions were studied in [17, 23]. A function $f \in \mathcal{S}$ is said to be starlike associated with hyperbolic tan function if the subordination inclusion $zf'(z)/f(z) \prec 1 + \tanh z$ holds or equivalently

$$\frac{zf'(z)}{f(z)} = 1 + \tanh(w(z)) \tag{8}$$

for all $z \in \mathbb{D}$, where w is an analytic function on $\mathbb{D}$ satisfying the inequality $|w(z)| < 1$ and condition $w(0) = 0$. The class of such functions is denoted by $S^*_{\tanh}$. Note that the image of unit disk $\mathbb{D}$ under the hyperbolic tangent function $1 + \tanh z$ is a bounded and starlike domain as shown in Figure 1.

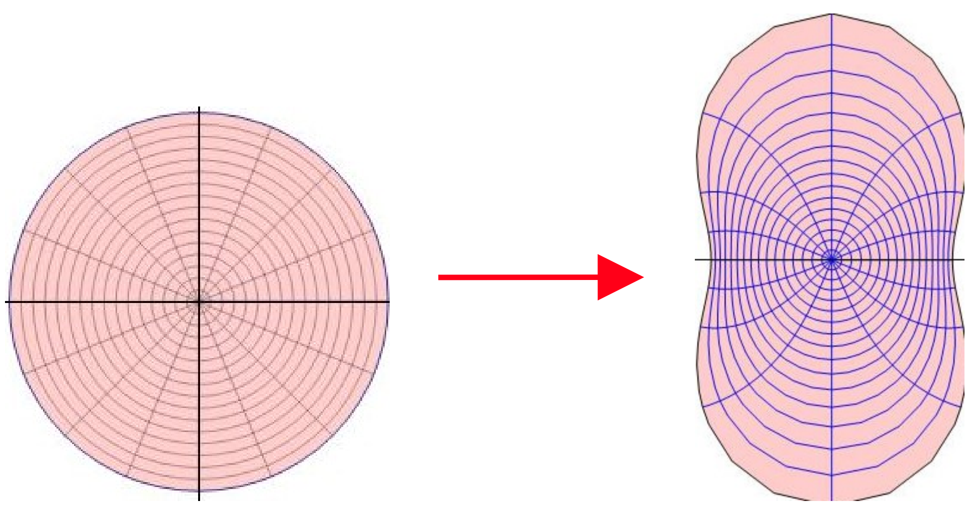

FIGURE 1. Image of unit disk $\mathbb{D}$ under the function $1 + \tanh z$

The study of this class was initiated by Ullah et al. [35] and the authors computed the radius of certain convexity, radius of Janowski starlikeness and some more radius constants of such functions. Further, authors [34] established best possible bounds on certain coefficient functionals like Fekete-Szegö and Hankel determinant of second and third order. Since $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots = (1 + w(z))/(1 - w(z))$, then using (8) we get

$$\frac{zf'(z)}{f(z)} = 1 + \tanh\left(\frac{p(z) - 1}{p(z) + 1}\right) \tag{9}$$

which implies

$$\begin{aligned}
&1 - a_2z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_2a_3 + 3a_4)z^3 + (-a_2^4 + 4a_2^2a_3 - 2a_3^2 - 4a_2a_4 \\
&\quad + 4a_5)z^4 + (a_2^5 - 5a_2^3a_3 + 5a_2^2a_4 + 5a_2a_3^2 - 5a_2a_5 - 5a_3a_4 + 5a_6)z^5 + (-a_2^6 \\
&\quad + 6a_2^4a_3 - 6a_2^3a_4 - 9a_2^2a_3^2 + 6a_2^2a_5 + 12a_2a_3a_4 - 6a_2a_6 + 2a_3^3 - 6a_3a_5 - 3a_4^2 \\
&\quad + 6a_7)z^6 + \cdots \\
&=1 + \frac{1}{2}p_1z + \frac{1}{4}\left(-p_1^2 + 2p_2\right)z^2 + \frac{1}{12}\left(p_1^3 - 6p_1p_2 + 6p_3\right)z^3 \\
&\quad + \frac{1}{4}\left(p_1^2p_2 - p_2^2 - 2p_1p_3 + 2p_4\right)z^4 + \frac{1}{4}(-13p_1^5 + 120p_1p_2^2 + 120p_1^2p_3 \\
&\quad - 240p_2p_3 - 240p_1p_4 + 240p_5)z^5 + \frac{1}{192}z^6(5p_1^6 - 26p_1^4p_2 + 48p_1^2p_4 \\
&\quad + 96p_1p_2p_3 - 96p_1p_5 + 16p_2^3 - 96p_2p_4 - 48p_3^2 + 96p_6) + \cdots .
\end{aligned}$$

On comparing the coefficients of like powers of z on both the sides, we get

$$a_2 = \frac{p_1}{2} \tag{10}$$

$$a_3 = \frac{p_2}{4} \tag{11}$$

$$a_4 = \frac{1}{72}(-p_1^3 - 3p_1p_2 + 12p_3) \tag{12}$$

$$a_5 = \frac{1}{576}(5p_1^4 - 12p_1^2p_2 - 18p_2^2 - 24p_1p_3 + 72p_4) \tag{13}$$

$$\begin{aligned}
a_6 = &\frac{1}{28800}(-111p_1^5 + 800p_1^3p_2 - 360p_1^2p_3 - 30p_1(7p_2^2 + 36p_4) \\
&+ 240(-7p_2p_3 + 12p_5))
\end{aligned} \tag{14}$$

$$\begin{aligned}
a_7 = &\frac{1}{518400}(626p_1^6 - 9375p_1^4p_2 + 11760p_1^3p_3 + 180p_1^2(97p_2^2 - 24p_4) - 360p_1(7p_2p_3 \\
&+ 48p_5) + 450(p_2^3 - 60p_2p_4 - 32p_3^2 + 96p_6)).
\end{aligned} \tag{15}$$

In [34], authors computed the best possible estimates on a_2, a_3, a_4, $H_2(2)(f)$ and $H_3(1)(f)$ for the functions $f \in S^*_{\tanh}$ which are given as:

Theorem 1.4. [34] *Let the function $f \in S^*_{tanh}$. Then*

$$|a_2| \leq 1, \quad |a_3| \leq \frac{1}{2}, \quad |a_4| \leq \frac{1}{3},$$

$$|H_2(2)(f)| \leq \frac{1}{4} \quad \textit{and} \quad |H_3(1)(f)| \leq \frac{1}{9}.$$

Each of these bounds is sharp.

In this paper, we determine necessary and sufficient convolution conditions, sharp bounds on $HT_2(1)(f)$, $HT_3(1)(f)$ for the functions $f \in S^*_{\tanh}$. Further, we compute estimates on the initial coefficients a_5, a_6, a_7, on the Hankel determinants $H_2(3)(f)$,

$H_3(2)(f)$, $H_4(1)(f)$ for such functions. In addition, using the concept of Briot-Bouquet differential subordination, we establish a subordination inclusion involving Bernardi integral operator.

2. **Convolution properties.** In view of [32, Theorem 1, p. 32], we have following necessary and sufficient convolution condition of the subclass $\mathcal{S}^*_{\tanh}$.

Theorem 2.1. *The function $f \in \mathcal{S}^*_{\tanh}$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - Lz^2}{(1-z)^2}\right] \neq 0 \tag{16}$$

for all $L = \frac{1+\tanh(e^{i\theta})}{\tanh(e^{i\theta})}$, where $0 \leq \theta \leq 2\pi$ and $L = 1$.

The coefficient estimate for a function f to be in the class $\mathcal{S}^*_{\tanh}$ has been computed in the next result.

Theorem 2.2. *If the function f satisfies the following inequality*

$$\sum_{n=2}^{\infty}\left|\frac{n - (1+\tanh(e^{i\theta}))}{\tanh(e^{i\theta})}\right||a_n| \leq 1, \tag{17}$$

then $f \in \mathcal{S}^*_{\tanh}$.

Proof. From Theorem 2.1, $f \in \mathcal{S}^*_{\tanh}$ if and only if

$$\frac{1}{z}\left[f(z) * \frac{z - Lz^2}{(1-z)^2}\right] \neq 0 \tag{18}$$

for all $L = \frac{1+\tanh(e^{i\theta})}{\tanh(e^{i\theta})}$ and also $L = 1$. The left-hand side of (18) is written as

$$\begin{aligned}
\frac{1}{z}\left[f(z) * \left(\frac{z}{(1-z)^2} - \frac{Lz^2}{(1-z)^2}\right)\right] &= \frac{1}{z}\big\{zf'(z) - L(zf'(z) - f(z))\big\} \\
&= 1 - \sum_{n=2}^{\infty}(n(L-1) - L)a_n z^{n-1} \\
&= 1 - \sum_{n=2}^{\infty}\frac{n-(1+\tanh(e^{i\theta}))}{\tanh(e^{i\theta})}a_n z^{n-1} \neq 0, \quad (19)
\end{aligned}$$

In view of the expression(19), we have

$$\begin{aligned}
\left|1 - \sum_{n=2}^{\infty}(n(L-1)-L)a_n z^{n-1}\right| &\geq 1 - \sum_{n=2}^{\infty}\left|(n(L-1)-L)a_n z^{n-1}\right| \\
&= 1 - \sum_{n=2}^{\infty}|(n(L-1)-L)||a_n||z|^{n-1} \\
&\geq 1 - \sum_{n=2}^{\infty}|(n(L-1)-L)||a_n| \\
&= 1 - \sum_{n=2}^{\infty}\left|\frac{n-(1+\tanh(e^{i\theta}))}{\tanh(e^{i\theta})}\right||a_n| \geq 0,
\end{aligned}$$

if the inequality (17) holds. ∎

3. **Hermitian-Toeplitz and Hankel determinants.** The sharp lower and upper bounds on the second and third-order Hermitian–Toeplitz determinants are computed for the starlike and convex functions of order α, close-to-star functions and the starlike function associated with Nephroid shaped domain [10,18,21]. Recently, authors [22] computed sharp estimates on $HT_3(1)(f)$ for the Sakaguchi classes and some of its associated subclasses. Here, we determine the sharp lower and upper bounds on $HT_2(1)$ and $HT_3(1)$ for the functions to be in the class $S^*_{\tanh}$.

Theorem 3.1. *Let the function $f \in \mathcal{S}^*_{tanh}$. Then*

$$0 \leq HT_2(1)(f) \leq 1 \quad \text{and} \quad -\frac{1}{4} \leq HT_3(1)(f) \leq 1.$$

Each of these bounds is sharp.

Proof. On substituting the value of a_2 from (10) in the expression of $HT_2(1)(f)$, we have $HT_2(1)(f) = 1 - \frac{p_1^2}{4}$. Since the class $S^*_{\tanh}$ and the second Hermitian determinant $HT_2(1)(f)$ are rotationally invariant, then $p_1 = x \in [0, 2]$. Therefore, $HT_2(1)(f) = 1 - \frac{x^2}{4}$ which has maximum value 1 and minimum value 0. The upper bound is sharp for the function f_0 defined as $zf_0'(z)/f_0(z) = 1 + \tanh z^2$ or

$$f_0(z) = z + \frac{1}{2}z^3 + \dots. \tag{20}$$

The lower bound is sharp for the function f_1 given as $zf_1'(z)/f_1(z) = 1 + \tanh z$ or

$$f_1(z) = z + z^2 + \frac{1}{2}z^3 + \dots. \tag{21}$$

On substituting the value of a_2 and a_3 from (10) and (11) respectively in the expression of $HT_3(1)(f)$, we have

$$HT_3(1)(f) = \frac{1}{16}\mathrm{Re}(p_1^2.2\bar{p_2}) - \frac{1}{2}|p_1|^2 - \frac{1}{16}|p_2|^2 + 1. \tag{22}$$

By making use of Lemma (1.1), we have

$$\mathrm{Re}(p_1^2.2\bar{p_2}) = \mathrm{Re}(p_1^2(p_1^2 + (4 - p_1^2)\bar{\xi})) = [p_1^4 + p_1^2(4 - p_1^2)\mathrm{Re}(\bar{\xi})], \tag{23}$$

and

$$\begin{aligned} |p_2|^2 = \frac{1}{4}|2p_2|^2 =& \frac{1}{4}|p_1^2 + (4 - p_1^2)\xi|^2 \\ =& -\frac{1}{4}[p_1^4 + (4 - p_1^2)^2|\xi|^2 + 2p_1^2(4 - p_1^2(4 - p_1^2)\mathrm{Re}(\bar{\xi})] \end{aligned} \tag{24}$$

for some $\xi \in \overline{\mathbb{D}}$. Thus, in view of (22), (23) and (24), we have

$$\begin{aligned} T_{3,1}(f) &= \frac{3}{64}p_1^4 - \frac{1}{64}(4 - p_1^2)^2|\xi|^2 + \frac{1}{32}p_1^2(4 - p_1^2)\mathrm{Re}(\bar{\xi}) - \frac{1}{2}p_1^2 + 1 \\ &= Y(p_1^2,\ |\xi|,\ \mathrm{Re}(\bar{\xi})). \end{aligned} \tag{25}$$

Note that $T_{3,1}(f) \leq Y(p_1^2, |\xi|, |\xi|) =: \boldsymbol{U}_1(p_1^2, |\xi|)$, and $T_{3,1}(f) \geq Y(p_1^2, |\xi|, -|\xi|) =: \boldsymbol{U}_2(p_1^2, |\xi|)$. With these attentions, and setting $p_1^2 = x \in [0, 4]$ and $|\xi| = y \in [0, 1]$, from here we have $\boldsymbol{U}_1(x, y) = 1 + [3x^2 - (4 - x)^2y^2 + 2x(4 - x)y - 32x]/64$ and $\boldsymbol{U}_2(x, y) = 1 + [3x^2 - (4 - x)^2y^2 - 2x(4 - x)y - 32x]/64$. In the rectangular region $[0, 4] \times [0, 1]$, the function $\boldsymbol{U}_1(x, y)$ has maximum value 1 and the function $\boldsymbol{U}_2(x, y)$ has minimum value $-\frac{1}{4}$ which completes the desired proof. The lower bound on $T_{3,1}(f)$ is the best possible for the function f_1 defined by (21). From the expansion of function f_1, we have $a_2 = 1,\ a_3 = \frac{1}{2}$. This immediately shows that

$T_3(1) := 2\Re(a_2^2\,\bar{a_3}) - 2|a_2|^2 - |a_3|^2 + 1 = -\frac{1}{4}$, highlighting the sharpness of the lower bound. □

In [31], the bounds on Hankel determinants associated with the coefficients in the series expansion of the functions $f \in \mathcal{S}$ were discussed. In [4], bounds on $H_3(1)(f)$ were computed for certain analytic functions as well as starlike and convex functions. Kowalczyk *et al.* [19] established sharp bound 4/135 on $H_3^{(1)}(f)$ for convex functions. Lecko *et al.* [27] investigated sharp bound on $H_3^{(1)}(f)$ for starlike functions of order 1/2. In [9, 26], bounds on $H_4(1)(f)$ for certain Ma-Minda type starlike functions were discussed. For recent results related to Hankel determinants, see [27, 36]. Next results provide estimates on initial coefficients a_5, a_6, a_7, second and third Hankel determinants $H_2(3)(f)$, $H_3(2)(f)$ and some coefficient functionals $a_3a_6 - a_4a_5$, $a_4a_6 - a_5^2$, $a_3a_5 - a_2a_6$ which are used in the determination of estimate on fourth Hankel determinant $H_4(1)(f)$ for $f \in S^*_{\tanh}$.

Theorem 3.2. *Let the function $f \in S^*_{tanh}$. Then*

(a) $|a_5| \leq \frac{13}{24}$, $\quad |a_6| \leq \frac{289}{600} + \frac{7}{10}\sqrt{\frac{7}{33}} \approx 0.804063$, $\quad |a_7| \leq \frac{791}{540} + \frac{2}{3}\sqrt{\frac{5}{33}} \approx 1.72431.$
(b) $|a_3a_6 - a_4a_5| \leq \frac{727}{1350} \approx 0.538519.$
(c) $|a_4a_6 - a_5^2| \leq \frac{77789}{129600} + \frac{11}{12\sqrt{210}} \approx 0.66348.$
(d) $|H_2(3)(f)| = |a_3a_5 - a_4^2| \leq \frac{1}{270}(65 + 6\sqrt{30}) \approx 0.362457.$
(e) $|a_3a_5 - a_2a_6| \leq \frac{19}{20} \approx 0.95.$

Proof. (a) On rearranging the terms in expressions (13), (14) and (15), we get

$$\begin{aligned}
576a_5 &= 12p_1^2\alpha_1 + 72\alpha_2 - 18p_2^2,\\
28800a_6 &= 800p_2\alpha_3 + 2880\alpha_4 - 111p_1^5 - 210p_1p_2^2 - 360p_1^2p_3,\\
518400a_7 &= 9375p_1^4\alpha_5 + 4320p_1^2\alpha_6 + 2700\alpha_7 + 11760\alpha_8 + 43200\alpha_9 - 2520p_1p_2p_3,
\end{aligned}$$

where

$$\alpha_1 = \frac{5}{12}p_1^2 - p_2,\ \alpha_2 = -\frac{1}{3}p_1p_3 + p_4,\ \alpha_3 = p_1^3 - \frac{21}{10}p_3$$
$$\alpha_4 = -\frac{3}{8}p_1p_4 + p_5,\ \alpha_5 = \frac{626}{9375}p_1^2 - p_2,\ \alpha_6 = \frac{97}{24}p_2^2 - p_4,$$
$$\alpha_7 = \frac{1}{60}p_2^2 - p_4,\ \alpha_8 = p_1^3 - \frac{60}{49}p_3,\ \alpha_9 = -\frac{2}{5}p_1p_5 + p_6.$$

On applying triangle inequality, we have

$$576|a_5| \leq 12|p_1|^2|\alpha_1| + 72|\alpha_2| + 18|p_2^2|, \tag{26}$$
$$28800|a_6| \leq 800|p_2||\alpha_3| + 2880|\alpha_4| + 111|p_1|^5 + 210|p_1||p_2|^2 - 360|p_1|^2|p_3|, \tag{27}$$
$$\begin{aligned}518400|a_7| \leq{}& 9375|p_1|^4|\alpha_5| + 4320|p_1|^2|\alpha_6| + 2700|\alpha_7| + 11760|\alpha_8| + 43200|\alpha_9|\\ &+ 2520|p_1||p_2||p_3|,\end{aligned} \tag{28}$$

On applying Lemmas 1.2 and 1.3, we have $|\alpha_1| \leq 2$, $|\alpha_2| \leq 2$, $|\alpha_3| \leq 21\sqrt{21}/5\sqrt{11}$, $|\alpha_4| \leq 2$, $|\alpha_5| \leq 2$, $|\alpha_6| \leq 85/6$, $|\alpha_7| \leq 2$, $|\alpha_8| \leq \frac{240}{49}\sqrt{\frac{15}{11}}$ and $|\alpha_9| \leq 2$. On substituting the estimates on $|\alpha_i|$; $i = 1, 2\cdots 9$ and using the inequality $|p_i| \leq 2$; $i \geq 1$ in the inequalities (26), (27) and (28), we get the desired estimates on a_5, a_6 and a_7.

(b) On putting the values of a_3, a_4, a_5 and a_6 from (11), (12), (13) and (14) respectively in $a_3a_6 - a_4a_5$, we get

$$\begin{aligned}
&1036800(a_3a_6 - a_4a_5)\\
=&125p_1^7 - 924p_1^5p_2 - 2100p_1^4p_3 + 450p_1^3(13p_2^2 + 4p_4) - 1440p_1^2p_2p_3\\
&- 360p_1(9p_2^3 + 12p_2p_4 - 20p_3^2) - 1080(9p_2^2p_3 - 24p_2p_5 + 20p_3p_4)\\
=&125p_1^7 - 924p_1^5p_2 - 2100p_1^4p_3 + 5850p_1^3p_2^2 + 1800p_1^3p_4 - 1440p_1^2p_2p_3\\
&- 3240p_1p_2^3 - 4320p_1p_2p_4 + 7200p_1p_3^2 - 9720p_2^2p_3\\
&+ 25920p_2p_5 - 21600p_3p_4
\end{aligned}$$

so that

$$\begin{aligned}
&1036800|a_3a_6 - a_4a_5|\\
\leq&924|p_1|^5|\beta_1| + 3240|p_1||p_2|^2|\beta_2| + 1800|p_1|^3|\beta_3| + 7200|p_1||p_3||\beta_4|\\
&+ 25920|p_2||\beta_5| + 9720|p_2|^2|p_3| + 21600|p_3||p_4| \qquad (29)
\end{aligned}$$

where

$$\beta_1 = \frac{125}{924}p_1^2 - p_2,\ \beta_2 = \frac{65}{36}p_1^2 - p_2,\ \beta_3 = -\frac{7}{6}p_1p_3 + p_4$$
$$\beta_4 = -\frac{1}{5}p_1p_2 + p_3,\ \beta_5 = -\frac{1}{6}p_1p_4 + p_5.$$

Using Lemma 1.2, we have $|\beta_1| \leq 2$, $|\beta_2| \leq \frac{47}{9}$, $|\beta_3| \leq \frac{8}{3}$, $|\beta_4| \leq 2$ and $|\beta_5| \leq 2$. On substituting the estimates on $|\beta_i|$; $i = 1, 2, 3, 4, 5$ and using the inequality $|p_i| \leq 2$; $i \geq 1$ in the inequality (29), we get $1036800|a_3a_6 - a_4a_5| \leq 558336$ which gives the desired estimate on $|a_3a_6 - a_4a_5|$.

(c) On putting the values of a_4, a_5 and a_6 from (12), (13) and (14) respectively in $a_4a_6 - a_5^2$, we get

$$\begin{aligned}
&8294400(a_4a_6 - a_5^2)\\
=&4(p_1^3 + 3p_1p_2 - 12p_3)(111p_1^5 - 800p_1^3p_2 + 360p_1^2p_3 + 30p_1(7p_2^2 + 36p_4)\\
&+ 240(7p_2p_3 - 12p_5)) - 25(-5p_1^4 + 12p_1^2p_2 + 24p_1p_3 + 18(p_2^2 - 4p_4))^2\\
=&- 181p_1^8 + 1132p_1^6p_2 + 2112p_1^5p_3 - 7860p_1^4p_2^2 - 13680p_1^4p_4\\
&+ 35040p_1^3p_2p_3 - 11520p_1^3p_5 - 8280p_1^2p_2^3 + 56160p_1^2p_2p_4\\
&- 31680p_1^2p_3^2 - 11520p_1p_2^2p_3 - 34560p_1p_2p_5 + 34560p_1p_3p_4 - 8100p_2^4\\
&+ 64800p_2^2p_4 - 80640p_2p_3^2 + 138240p_3p_5 - 129600p_4^2
\end{aligned}$$

which implies that

$$\begin{aligned}
&8294400|a_4a_6 - a_5^2|\\
\leq&1132|p_1|^6|\beta_6| + 35040|p_1|^3|p_2||\beta_7| + 2112|p_3||p_1|^2|\beta_8| + 64800|p_2|^2|\beta_9|\\
&+ 56160|p_1|^2|p_4||\beta_{10}| + 129600|p_4||\beta_{11}| + 138240|p_5||\beta_{12}|\\
&+ 8280|p_1|^2|p_2|^3 + 8100|p_2|^4 + 80640|p_2||p_3|^2 + 11520|p_1|^3|p_5| \qquad (30)
\end{aligned}$$

where

$$\beta_6 = -\frac{181}{1132}p_1^2 + p_2,\ \beta_7 = -\frac{131}{584}p_1p_2 + p_3,\ \beta_8 = p_1^3 - 15p_3,$$
$$\beta_9 = -\frac{8}{45}p_1p_3 + p_4,\ \beta_{10} = -\frac{19}{78}p_1^2 + p_2,\ \beta_{11} = \frac{4}{15}p_1p_3 - p_4,\ \text{and}\ \beta_{12} = \frac{1}{4}p_1p_2 - p_3.$$

By making use of Lemmas 1.2 and 1.3, we have $|\beta_6| \leq 2$, $|\beta_7| \leq 2$, $|\beta_8| \leq 30\sqrt{15/14}$, $|\beta_9| \leq 2$, $|\beta_{10}| \leq 2$, $|\beta_{11}| \leq 2$, and $|\beta_{12}| \leq 2$. On substituting the estimates on $|\beta_i|$; $i = 6, 7, \cdots, 12$ and using $|p_i| \leq 2$; $i \geq 2$ in the inequality (30), then we get the desired bound on $a_4a_6 - a_5^2$.

(d) On putting the values of a_3, a_4 and a_5 from (11), (12) and (13) respectively in $H_2(3)(f) = a_3a_5 - a_4^2$, we get

$$\begin{aligned}
&20736H_2(3)(f)\\
=& -4(p_1^3 + 3p_1p_2 - 12p_3)^2 - 9p_2(-5p_1^4 + 12p_1^2p_2 + 24p_1p_3 + 18(p_2^2 - 4p_4))\\
=& -4p_1^6 + 21p_1^4p_2 - 144p_1^2p_2^2 - 162p_2^3 + 96p_1^3p_3 + 72p_1p_2p_3 - 576p_3^2\\
&+ 648p_2p_4
\end{aligned}$$

so that

$$20736|H_2(3)(f)| \leq 21|p_1|^4|\beta_{13}| + 72|p_1||p_2||\beta_{14}| + 648|p_2||\beta_{15}| + 96|p_3||\beta_{16}| \tag{31}$$

where

$$\beta_{13} = -\frac{4}{21}p_1^2 + p_2,\ \beta_{14} = -2p_1p_2 + p_3,\ \beta_{15} = -\frac{1}{4}p_2^2 + p_4,\ \beta_{16} = p_1^3 - 6p_3.$$

Making use of Lemmas 1.2 and 1.3, we have $|\beta_{13}| \leq 2$, $|\beta_{14}| \leq 6$, $|\beta_{15}| \leq 2$ and $|\beta_{16}| \leq 12\sqrt{6/5}$. On putting the estimates on $|\beta_i|$; $i = 13, 14, 15, 16$ and using $|p_i| \leq 2$; $i \geq 2$ in the inequality (31), we get the desired bound on $a_4a_6 - a_5^2$.

(e) On putting the values of a_2, a_3, a_5 and a_6 from (10), (11) (13) and (14) respectively in $a_3a_5 - a_2a_6$, we get

$$\begin{aligned}
&19200(a_3a_5 - a_2a_6)\\
=&37p_1^6 - 225p_1^4p_2 + 120p_1^3p_3 - 30p_1^2(p_2^2 - 12p_4) - 150(p_2^3 - 4p_2p_4)\\
&+ 120p_1(3p_2p_3 - 8p_5)\\
=&37p_1^6 - 225p_1^4p_2 - 30p_1^2p_2^2 - 150p_2^3 + 120p_1^3p_3 + 360p_1p_2p_3\\
&+ 360p_1^2p_4 + 600p_2p_4 - 960p_1p_5
\end{aligned}$$

so that

$$\begin{aligned}
19200|a_3a_5 - a_2a_6| \leq& 225|p_1|^4|\beta_{17}| + 360|p_1||p_2||\beta_{18}| + 600|p_2||\beta_{19}|\\
&+ 960|p_1||\beta_{20}| + 120|p_1|^3|p_3|
\end{aligned} \tag{32}$$

where

$$\beta_{17} = \frac{37}{225}p_1^2 - p_2,\ \beta_{18} = -\frac{1}{12}p_1p_2 + p_3,\ \beta_{19} = -\frac{1}{4}p_2^2 + p_4,\ \beta_{20} = \frac{3}{8}p_1p_4 - p_5.$$

Using Lemmas 1.2 and 1.3, we get $|\beta_{17}| \leq 2$, $|\beta_{18}| \leq 2$, $|\beta_{19}| \leq 2$ and $|\beta_{20}| \leq 2$. On putting the estimates on $|\beta_i|$; $i = 17, 18, 19, 20$ and using $|p_i| \leq 2$; $i \geq 2$ in the inequality (32), then we get the desired bound on $a_3a_5 - a_2a_6$. □

Theorem 3.3. *Let the function $f \in S^*_{tanh}$. The coefficient functionals $H_3(2)(f)$, δ_1 and δ_2 are given in the expressions (3) and (4) respectively. Then*

$$|H_3(2)(f)| \leq \frac{172319 + 1344\sqrt{30} + 792\sqrt{210}}{181440} \approx 1.05356$$

$$|\delta_1| \leq \frac{3731}{2700} \approx 1.38185 \text{ and } |\delta_2| \leq \frac{163181}{129600} + \frac{11}{12\sqrt{210}} + \frac{1}{3\sqrt{30}} \approx 1.38323$$

Proof. By applying triangle inequality in the expressions of $H_3(2)(f)$, δ_1, and δ_2, we get

$$|H_3(2)(f)| \leq |a_2||a_4a_6 - a_5^2| + |a_3||a_3a_6 - a_4a_5| + |a_4||H_2(3)(f)|, \tag{33}$$

$$|\delta_1| \leq |a_3a_6 - a_4a_5| + |a_2||a_3a_5a_2a_6| + |a_4||H_2(2)(f)| \tag{34}$$

$$|\delta_2| \leq |a_4a_6 - a_5^2| + |a_2||a_3a_6 - a_4a_5| + |a_3||H_2(3)(f)| \tag{35}$$

On substituting the estimates on a_2, a_3, a_4, $a_4a_6 - a_5^2$, $a_3a_6 - a_4a_5$, $H_2(3)(f)$, $H_2(2)(f)$, $a_3a_5 - a_2a_6$, from Theorem 1.4, and Theorem 3.2 in the inequalities (33), we get the desired bound on $H_3(2)(f)$, δ_1 and δ_2. □

Theorem 3.4. *Let the function $f \in S^*_{tanh}$. Then the bound on fourth order Hankel determinant $H_4(1)(f)$ is given as*

$$|H_4(1)(f)| \leq \frac{121000\sqrt{30}\left(20 + 9\sqrt{7}\right) + 640000\sqrt{165} + 8357440\sqrt{231} + 520898609}{285120000}.$$

Proof. On applying triangle inequality in the expressions (4) of the Hankel determinant $H_4(1)(f)$, we get

$$|H_4(1)(f)| \leq |a_7||H_3(1)(f)| + |a_6||\delta_1| + |a_5||\delta_2| + |a_4||H_3(2)(f)|.$$

On substituting the estimates on a_4, a_5, a_6, a_7, $H_3(1)(f)$, δ_1, δ_2 and $H_3(2)(f)$ from Theorem 1.4, Theorem 3.2 and Theorem 3.3, we get the desired approximate estimate 2.40312 on $|H_4(1)(f)|$ for $f \in S^*_{\tanh}$. □

4. **Application of Bernardi integral operator.** By using a technique based upon Briot-Bouquet differential subordination which was investigated by Miller and Mocanu [29], we establish the subordination property as below. In order to prove the result, following lemma is needed:

Lemma 4.1. *[16] Assume φ ($\varphi(0) = 1$) is convex univalent in $\mathbb{D}$. Also, assume ϕ of the form $\phi(z) = 1 + c_1z + c_2z^2 + \cdots$ ($\phi(0) = 1$) is analytic in $\mathbb{D}$. If*

$$\phi(z) + \frac{1}{\mu}z\phi'(z) \prec \varphi(z) \quad (\mu \neq 0,\ \Re\mu \geq 0),$$

then

$$\phi(z) \prec \tilde{h}(z) = \frac{\mu}{z^\mu}\int_0^z t^{\mu-1}\varphi(t)dt \prec \varphi(z) \tag{36}$$

and $\tilde{h}$ is the best dominant.

Theorem 4.2. *Let $0 < \lambda < 1$ and $\zeta \geq 1$. If the function $f \in \mathcal{A}$ satisfies following subordination condition*

$$(1-\lambda)\frac{f(z)}{z} + \lambda\frac{\mathcal{L}_\sigma f(z)}{z} \prec 1 + \tanh z, \tag{37}$$

then

$$\Re\left\{\left(\frac{\mathcal{L}_\sigma(z)}{z}\right)^{1/\zeta}\right\} > \left(\frac{1+\sigma}{1-\lambda}\int_0^1 u^{\frac{1+\sigma}{1-\lambda}-1}(1 - \tanh u \sec^2 u)du\right)^{1/\zeta}. \tag{38}$$

Proof. Consider the analytic function

$$\phi(z) = \frac{\mathcal{L}_\sigma f(z)}{z}; \qquad z \in \mathbb{D}$$

with $\phi(0)=1$ in $\mathbb{D}$. Differentiating both sides of this identity, and using (6), we get

$$\frac{f(z)}{z}=\phi(z)+\frac{1}{1+\sigma}z\phi'(z)$$

Applying (37), we conclude

$$(1-\lambda)\frac{f(z)}{z}+\lambda\frac{\mathcal{L}_{\sigma,p}f(z)}{z}=\phi(z)+\frac{1-\lambda}{1+\sigma}z\phi'(z)\prec 1+\tanh z.$$

Thus, follows from Lemma 4.1 that

$$\phi(z)\prec\frac{1+\sigma}{1-\lambda}z^{-\frac{1+\sigma}{1-\lambda}}\int_0^z t^{\frac{1+\sigma}{1-\lambda}-1}(1+\tanh t)dt$$

or

$$\frac{\mathcal{L}_{\sigma}f(z)}{z}=\frac{1+\sigma}{1-\lambda}\int_0^1 u^{\frac{1+\sigma}{1-\lambda}-1}(1+\tanh uw(z))du.$$

where w is a Schwarz function with $w(0)=0$ and $|w(z)|<1$ in $\mathbb{D}$. Since (see Ullah et.al [35])

$$\Re(1+\tanh w(z))>1-\tanh r\sec^2 r.$$

for $0<\lambda<1$, and letting $r\to 1^-$, we arrive at

$$\Re\left(\frac{\mathcal{L}_{\sigma}f(z)}{z}\right)>\frac{1+\sigma}{1-\lambda}\int_0^1 u^{\frac{1+\sigma}{1-\lambda}-1}(1-\tanh u\sec^2 u)du>0,\quad z\in\mathbb{D}. \tag{39}$$

Since $\Re(w^{1/\zeta})\geq\Re(w)^{1/\zeta}$ for $\Re(w)>0$ and $\zeta\geq 1$, from (39) we prove the inequality (38). $\square$

5. **Conclusion.** In this article, we have discussed some convolution related properties for the functions $f\in S^*_{\tanh}$. We also obtained the best possible lower and upper bounds on Hermitian-Toeplitz determinants for the classes $S^*_{\tanh}$. Furthermore, the method we have used for computation of estimate on certain coefficient functionals and some Hankel determinants for such functions is explained. In addition, a subordination inclusion involving Bernardi integral operator has established.

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