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SCHWARZIAN DERIVATIVE REVISITED

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ABSTRACT. The Boolean-algebraic structure of the so-called Schwarzian derivative is investigated. A sufficient condition for a function of several variables to behave chaotically, which concerns its associated Schwarzian derivative, is also given.

1. INTRODUCTION

It is almost a commonplace observation that as long as the word 'chaos' is considered as intimately related to mathematics, the picture that emerges for a variety of occurrences becomes quickly untenable, and mathematical statements about an enormously increasing amount of information in the steady stream of experimental phenomena get a vague character. The reason that lies behind this, to our belief, is the lack of a general theory which could combine and unify, at least to some reasonable extent, the diversity one comes up when one considers any situation which involves 'chaotic behaviour'. It should be clear that such an attempt to unifying at least the different definitions of the word 'chaos' will provide a solid base for the studies and a clear perspective for those who are dealing with them. It is also immediate that this is a rather difficult task, and the fact of paramount importance is that this should not be taken as freezing the vivid scientific background. To this circumstance, a good starting point could be looking at those phenomena that was/has been/are considered as 'strange' or 'mysterious' in the scientific community from a different point of view. A possible attempt in this direction, which concerns an important tool, namely the Schwarzian derivative of a map, constitutes the subject matter of this expository article.

2. HEURISTICS

The necessary ingredients for our goal are the definitions and the basic properties of the Schwarzian derivative, Feigenbaum constant, and Boolean algebra.

2.1. **Schwarzian derivative.** Let $\Omega \neq \mathbb{C}$ be a region (i.e., an open and connected set) in the complex plane $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be an analytic and locally univalent function. The function S_f defined by

$$S_f(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{2.1}$$

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is called the *Schwarzian derivative* of f. Undoubtedly, the most important property of S_f is its invariance under the action of the group of Möbius transformations: if

$$\mathscr{M} := \left\{ \mathfrak{r}(z) = \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{C},\ ad-bc \neq 0 \right\},$$

then a straightforward computation implies that

$$S_{\mathfrak{r}\circ f} = S_f$$

for all $\mathfrak{r} \in \mathscr{M}$. Moreover, for any function g that is analytic and locally univalent on the range of f, the chain-rule-like formula

$$S_{g\circ f} = (S_g \circ f)(f')^2 + S_f \tag{2.2}$$

holds true. Since $S_{\mathfrak{r}} = 0$ for all $\mathfrak{r} \in \mathscr{M}$, the equality

$$S_{g\circ\mathfrak{r}} = (S_g \circ \mathfrak{r})(\mathfrak{r}')^2$$

is obtained as a special case of (2.2). Further, for any function ψ that is analytic in a simply connected domain Ω in the complex plane, there exists a meromorphic function f in Ω such that

$$S_f = \psi,$$

and this solution is unique up to an arbitrary Möbius transformation. In particular, a function is determined by its Schwarzian derivative up to a Möbius transformation.

Note that the Schwarzian derivative S_f measures the deviation of f from a Möbius transformation in a similar spirit as the ordinary derivative of a function measures its deviation from a linear map. A thorough explanation of the Schwarzian derivative and its properties can be found in [2] and [5].

2.2. **Feigenbaum constant.** Consider the iterations of the function

$$f(x) = 1 - \mu|x|^r,$$

defined for the real values of x over the unit interval, as the bifurcation parameter μ is increased for some fixed x and $r > 1$. Denote by μ_n the point at which a period of 2^n-cycle appears. Then the limit

$$\delta := \lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}}$$

exists and is called the *Feigenbaum constant.* Moreover, if one lets d_n be the value of the nearest cycle element to zero in the 2^n-cycle, then the value

$$\alpha := \lim_{n\to\infty} \frac{d_n}{d_{n+1}},$$

where the appearing limit also exists, is called the *reduction parameter* associated to δ. It is well-known that the Feigenbaum constant δ and its associated reduction parameter α are universal for all one-dimensional maps having a single locally quadratic maximum. We refer to [6] for a more detailed explanation of the Feigenbaum constant.

2.2.1. The connection between the Schwarzian derivative S_f of a one-dimensional real-valued map and the Feigenbaum constant is that a sufficient condition for f to be a 'universal' map, i.e., a map for which the associated Feigenbaum constant is universal, is that $S_f(x) < 0$ for all real values of x in a bounded interval. When such a situation occurs, f is said to *behave chaotically.*

2.3. **Boolean algebras.** A *Boolean algebra* $\mathbb{B}$ is a ring $\langle B, +, \times, 0, 1\rangle$ in which each element is an idempotent for multiplication (i.e., is equal to its square). This is equivalent to saying that a Boolean algebra is a distributive complemented lattice with the usual lattice operations $\vee$ and $\wedge$ and with zero and unity. A Boolean algebra is said to be *complete* if each subset of it has a supremum and infimum. The set of homomorphisms of a Boolean algebra $\mathbb{B}$ into $\{0, 1\}$ is denoted by $S(\mathbb{B})$ and is called the *Stone space* of $\mathbb{B}$. $S(\mathbb{B})$ is a compact and zero-dimensional topological space with base $\{\mathcal{U}(b) \mid b \in \mathbb{B}\}$, where $\mathcal{U}(b)$ denotes the set of ultra-filters on $\mathbb{B}$ containing b. The celebrated *Stone's theorem* states that every Boolean algebra is isomorphic to the Boolean algebra of clopen subsets of its Stone space (cf. [1]).

Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$\mathbb{V}_{\alpha}^{(\mathbb{B})} := \{x \mid x \text{ is a function } \wedge\ (\exists\beta)(\beta < \alpha\ \wedge\ \mathrm{dom}(x) \subset \mathbb{V}_{\beta}^{(\mathbb{B})}\ \wedge\ \mathrm{im}(x) \subset \mathbb{B})\}.$$

By means of this recursive definition, the *Boolean-valued universe* $\mathbb{V}^{(\mathbb{B})}$ is defined by

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathcal{O}} \mathbb{V}_{\alpha}^{(\mathbb{B})},$$

where $\mathcal{O}$ stands for the class of all ordinals. In case of the two element Boolean algebra $\mathbf{2} := \{0, 1\}$, this procedure yields a version of the classical *von Neumann universe* $\mathbb{V}$.

Let φ be an arbitrary formula of ZFC, the Zermelo-Fraenkel set theory with the Axiom of Choice. The *Boolean truth value* $[\![\varphi]\!] \in \mathbb{B}$ is introduced by induction on the length of a formula φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra $\mathbb{B}$ and taking into consideration the way in which this formula is built up. The Boolean truth values of the *atomic formulas* $x \in y$ and $x = y$, with $x, y \in \mathbb{V}^{(\mathbb{B})}$, are defined via the following recursion schema:

$$[\![x \in y]\!] = \bigvee_{t \in \mathrm{dom}(y)} y(t) \wedge [\![t = x]\!],$$

$$[\![x = y]\!] = \bigvee_{t \in \mathrm{dom}(x)} x(t) \Rightarrow [\![t \in y]\!] \wedge \bigvee_{t \in \mathrm{dom}(y)} y(t) \Rightarrow [\![t \in x]\!].$$

The sign $\Rightarrow$ symbolizes the implication in $\mathbb{B}$; i.e., $a \Rightarrow b := a^* \vee b$, where a^* is the usual complement of a which is defined as an element of the lattice B such that $a \wedge a^* = 0$ and $a \vee a^* = 1$.

The universe $\mathbb{V}^{(\mathbb{B})}$ with the Boolean truth value of a formula is a model of set theory in the sense that the following statement is fulfilled.

TRANSFER PRINCIPLE. *For very theorem φ of* ZFC, *we have $[\![\varphi]\!] = 1$; i.e., φ is true inside $\mathbb{V}^{(\mathbb{B})}$.*

This statement is considered with the following agreement: if x is an element of $\mathbb{V}^{(\mathbb{B})}$ and $\varphi(\cdot)$ is a formula of ZFC, then the phrase "x satisfies φ inside $\mathbb{V}^{(\mathbb{B})}$" or, briefly, "$\varphi(x)$ is true inside $\mathbb{V}^{(\mathbb{B})}$" means that $[\![\varphi(x)]\!] = 1$. This is written as $\mathbb{V}^{(\mathbb{B})} \models \varphi(x)$.

MAXIMUM PRINCIPLE. *If φ is a formula of* ZFC, *then there is a $\mathbb{B}$-valued set x_0 satisfying $[\![(\exists x)\varphi(x)]\!] = [\![\varphi(x_0)]\!]$.*

An excellent treatment of the Boolean-valued analysis can be found in [3] and [4].

3. MAIN RESULTS

A natural generalization of the Schwarzian derivative of a map given by (2.1) can be interpreted as follows.

Definition 3.1. Let $\Omega \subset \mathbb{R}^n$ be a region and $f \in C^\infty(\Omega)$ with non-vanishing first partial derivatives, where $n > 1$. The function $S_f^{[x_i]}$ defined by

$$S_f^{[x_i]}(\mathrm{x}) := \frac{\partial}{\partial x_i}\left(\frac{f_{x_i x_i}(\mathrm{x})}{f_{x_i}(\mathrm{x})}\right) - \frac{1}{2}\left(\frac{f_{x_i x_i}(\mathrm{x})}{f_{x_i}(\mathrm{x})}\right)^2,$$

where $\mathrm{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $i = 1, 2, \ldots, n$, is called the *partial Schwarzian derivative of f with respect to x_i*. Let now $2 \leq k \leq n$ and $(i_1, i_2, \ldots, i_k)$ be an ordered k-tuple consisting of a (possibly repeated) k-element permutation of the set $\{1, 2, \ldots, n\}$. The *mixed partial Schwarzian derivative of f of order k with respect to $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$* is defined recursively as

$$S_f^{[x_{i_1} x_{i_2} \cdots x_{i_k}]} := S_{f_{i_{k-1}}}^{[x_{i_k}]}, \quad \text{where} \quad f_{i_{k-1}} := S_f^{[x_{i_1} x_{i_2} \cdots x_{i_{k-1}}]}. \tag{3.1}$$

Moreover, for each $m = 2, 3, \ldots, n$, put

$$S_f^{[x_i^m]} := S_f^{[\overbrace{x_i x_i \cdots x_i}^{m \text{ times}}]} \tag{3.2}$$

for all $i \in \{1, 2, \ldots, n\}$.

Observe that the remark in 2.2.1 coupled with (2.1) yields the following fact: if for a one-dimensional real-valued map f of class $C^3(\Omega)$ (where Ω is an open set in the real line) having a non-vanishing first derivative, f' and f''' have different signs over Ω, then $S_f(x) < 0$ for all $x \in \Omega$, i.e., f behaves chaotically. To the best of our knowledge, it is still not known whether an analogue of the chaotic behaviour concept is meaningful for an n-dimensional map, where $n > 1$; but if it is, in the light of what one has just been observed, we have the following result which rests mainly on this simple observation.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a region and $f \in C^\infty(\Omega)$ with non-vanishing first partial derivatives, where $n > 1$. Suppose, moreover, that f behaves chaotically whenever $S_{f_{i_{k-1}}}^{[x_{i_k}]}$, which is defined in (3.1), is negative for a fixed $k \leq n$. Then a sufficient condition for this to happen is that all partial Schwarzian derivatives of f up to k be negative.*

Proof. Apply the recursion formula (3.1) successively to f and consider the obtained derivatives. It becomes then obvious to get the desired result. □

Remark 3.3. The condition of f to have negative partial Schwarzian derivatives up to k be negative in Theorem 3.2 can be weakened by appropriately eliminating those derivatives that do not have any effect on $S_{f_{i_{k-1}}}^{[x_{i_k}]}$. Unfortunately, as we do not have any useful information about the intrinsic properties of such a function yet, it seems just a mere computation to deal with it at this step.

3.1. Schwarzian derivative as a Boolean algebra. Let n be a positive integer and $1 \leq k \leq n$. Denote by P_k the set of all all ordered k-tuples consisting of a (possibly repeated) k-element permutation of the set $\{1, 2, \ldots, n\}$. Consider the set

$$P := \bigcup_{k=1}^{n} P_k.$$

Then, by abusing the notation and denoting again by f the underlying function f considered on the corresponding hyperplane, the identity (3.2) implies that the symbolic operator $S_f^{[x_i]}$ acts idempotently to each free variable of f; i.e., it is a Boolean algebra. The question comes then naturally to know what the structure of this Boolean algabra is, and what happens to the chaotic behaviour of a function if it is considered on a Boolean-valued universe defined as in 2.3. Moreover, the Feigenbaum constant needs also to be interpreted. Nothing is known about those facts yet, but it seems likely that information as such will shed light on the so-called chaotic behaviour of a function. But before all these, a more fundamental problem waits for an answer; namely, whether this Boolean algebra is complete or not.

4. CONCLUSIONS

The ideas presented in the present paper are quite new and the interpretations about the Boolean-valued analysis of the Schwarzian derivative is still an ongoing project. It has also become clear to the authors that such a system, namely a dynamical system considered on a Boolean-valued universe is not easy-to-deal-with at all. But the very rich structure of the Boolean-valued universes and their interpretations given in [3] allow one to infer that it is worth studying it.

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