

DYNAMICS ON RELAXED NEWTON'S METHOD DERIVATIVE

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ABSTRACT

In the present report the dynamic behaviour of the one dimensional family of maps $f(x) = b(x+a)^{-\lambda}$ is examined, for representative values of the control parameters a , b and λ . These maps are of special interest, since they are solutions of $N'_{\lambda f} = 2$, where $N'_{\lambda f}$ is the Relaxed Newton's method derivative. The maps $f(x)$ are proved to be solutions of the non-linear differential equation, $\frac{df(x)}{dx} - \beta \cdot [f(x)]^{(1+\lambda)/\lambda}$, where $\beta = \lambda \cdot b^{-1/\lambda}$. The recurrent form of these maps, $x_n = b(x_{n-1} + a)^{-\lambda}$, after excessive iterations, shows in a x_n vs. λ plot, an initial exponential decay followed by a bifurcation. The value of λ at which this bifurcation takes place, depends on the values of the parameters a , b . This corresponds to a switch to an oscillatory behaviour with amplitudes of $f(x)$ undergoing a period doubling. For values of a slightly higher than 1 and at higher λ 's a reverse bifurcation occurs and a bleb is formed. This behaviour is confirmed by calculating the corresponding Lyapunov exponent.

1. Introduction

The study of the derivatives of Newton's method and Relaxed Newton's method has attracted the attention of several groups involved in dynamical systems over the last years. In fact, due to its iterative schemes, discrete dynamics techniques, have disclosed striking results in the area of chaotic dynamics [1,3,5]. In most cases, these methods and their derivatives have been applied on specific maps in the real and complex plane [2,10]. Nevertheless, it is surprisingly interesting to view such iterative schemes, serving as dynamical systems by themselves.

In previous reports [7,8], we have studied the cases where the derivative of Newton's method takes on specific values, namely in $(0,2)$. The choice of these values has been motivated both by preliminary numerical results on electronic circuits and the incorporation of the Schwarzian derivative in our studies. The Schwarzian derivative, which was defined by the German mathematician Hermann Schwarz in 1869 for studying complex valued functions and has been essentially used in dynamics since 1978 [9], offers conditions under which such systems can be driven to chaos via period doubling bifurcations.

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2. Relaxed Newton's Method Derivative

For any real- or complex-valued function f , we define the *Newton Method* on f , by

$$N_f(x) = x - \frac{f(x)}{f'(x)} \quad (1)$$

Because Newton's method is only linearly convergent at multiple roots, various modifications have been suggested for improving the convergence. When we know that f

has a multiple root of order λ , we can apply Newton's method to $f(x)^{\frac{1}{\lambda}}$, to obtain

$$N_{\lambda,f} = x - \frac{f(x)^{\frac{1}{\lambda}}}{(1/\lambda)f(x)^{\frac{1}{\lambda}-1}f'(x)} = x - \lambda \cdot \frac{f(x)}{f'(x)} \quad (2)$$

This is called *Relaxed Newton Method* or *Newton's method for a root of order λ* . This method converges quadratically to a root of order exactly λ .

One can define the first derivative of f by both the Newton's method and the Relaxed Newton's method, namely

$$N'_f(x) = \left(x - \frac{f(x)}{f'(x)} \right)' = \frac{f(x)f''(x)}{(f'(x))^2} \quad (3)$$

and

$$N'_{\lambda,f}(x) = \left(x - \lambda \cdot \frac{f(x)}{f'(x)} \right)' = 1 - \lambda + \lambda \cdot \frac{f(x)f''(x)}{(f'(x))^2} \quad (4)$$

In previous works [7,8], we have examined the dynamic behaviour of the maps satisfying $N'_f(x) = 2$ and $N'_f(x) = a$, for $0 < a < 2$ and $a \neq 1$. We now generalize the above investigation to maps satisfying

$$N'_{\lambda,f}(x) = 2 \quad (5)$$

Simple calculations (based on successive integrations) show that $N'_{\lambda,f} = 2$ if and only if f is of the form

$$f(x) = b(x+a)^{-\lambda} \quad (6)$$

On the other hand,

$$N'_{\lambda,f}(x) = \left(x - \lambda \cdot \frac{f(x)}{f'(x)} \right)' = 1 - \lambda + \lambda \cdot \frac{f(x)f''(x)}{(f'(x))^2} = 2 \Rightarrow$$

$$\left[\frac{(1-\lambda) \frac{f'(x)}{f(x)} + \lambda \frac{f''(x)}{f'(x)}}{\frac{f'(x)}{f(x)}} \right] = 2 \Rightarrow$$

$$(1-\lambda)\frac{f'(x)}{f(x)} + \lambda\frac{f''(x)}{f'(x)} = 2\frac{f'(x)}{f(x)}$$

which can also be written as

$$\frac{f''(x)}{f'(x)} = \left(\frac{1+\lambda}{\lambda}\right)\frac{f'(x)}{f(x)}.$$

If both sides of this equation are integrated, then we obtain the following equation

$$f'(x) - \beta [f(x)]^{\left(\frac{1+\lambda}{\lambda}\right)} = 0 \quad (7)$$

The integral parameter β is calculated by substitution of (6) in (7):

$$\begin{aligned} -\frac{\lambda b}{(x+a)^{1+\lambda}} &= \beta \left(\frac{b}{(x+a)^\lambda}\right)^{1+\frac{1}{\lambda}} \Rightarrow -\frac{\lambda b}{(x+a)^{1+\lambda}} = \beta \left(\frac{b^{1+\frac{1}{\lambda}}}{(x+a)^{1+\lambda}}\right) \\ -\lambda b &= \beta b^{1+\frac{1}{\lambda}} \Rightarrow \frac{-\lambda}{b^{\left(\frac{1+\frac{1}{\lambda}}{\lambda}\right)-1}} = \beta \end{aligned}$$

Therefore: $\beta = \frac{-\lambda}{b^{\frac{1}{\lambda}}}$ (8)

Finally the differential equation given in (7) becomes

$$f'(x) + \frac{\lambda}{b^{\frac{1}{\lambda}}} [f(x)]^{\left(\frac{1+\lambda}{\lambda}\right)} = 0 \quad (9)$$

At this point, we incorporate in our discussion the idea of the *Schwarzian derivative*. Recall that the Schwarzian derivative of f is

$$S_f = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'}\right)^2 \quad (10)$$

It has to be mentioned here that the existence of period doublings is allowed but not guaranteed by the possession of a negative Schwarzian derivative S_f . Using expression (4) and after some elementary calculations we obtain the explicit expression for S_f :

$$f'(x) = -b\lambda(x+a)^{-(1+\lambda)} \quad (11)$$

$$f''(x) = b\lambda(1+\lambda)(x+a)^{-(2+\lambda)} \quad (12)$$

$$f'''(x) = -b\lambda(1 + \lambda)(2 + \lambda)(x + a)^{-(3+\lambda)} \quad (13)$$

Finally:

$$S_f = \frac{1 - \lambda^2}{2(x + a)^2} \quad (14)$$

Obviously, the sign of S_f depends on the values that λ takes on. In the following, we restrict our attention to the case $-1 < \lambda < 1$, where $S_f < 0$.

3. Exhibition of the Dynamics - Results

We examine the chaotic behavior of the family of maps

$$f(x) = b(x + a)^{-\lambda} \quad (15)$$

In a first step we fix the values a and b equal to 1 and we vary the parameter λ , in the range (0,30).

Iterating the recurrent form of (15):

$$x_n = b(x_{n-1} + a)^{-\lambda} \quad (16)$$

we obtain the curve of Figure 1. To avoid initial fluctuations we performed the averaging over the last 100 values of 10 000 iterations. For this purpose we have used Mathcad [4,6] to calculate the bifurcation diagrams. As it is evident from this curve, a bifurcation (period doubling) occurs at $\lambda \approx 4.12$.

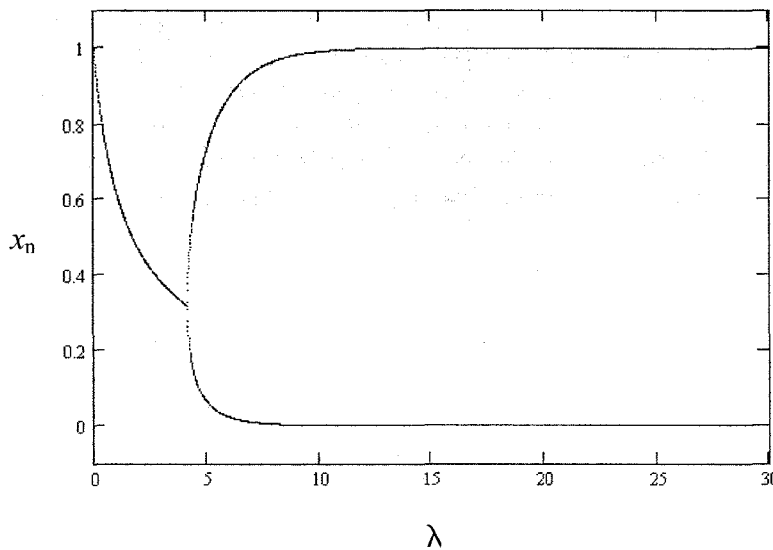


Figure 1

The curve obtained after iterating eq.(16) for $a=1$ and $b=1$

Increasing slightly the value of a , the shape of the curve undergoes the following changes:

- (i) a reverse bifurcation occurs at high values of λ . This way a bleb is formed.
- (ii) the critical values of λ at which the initial- and the reverse- bifurcations occur depend on the value of the parameter a . By increasing a , the critical values of λ shift to opposite directions, confining the size of the bleb, which finally disappears. This behaviour is illustrated in Figure 2.

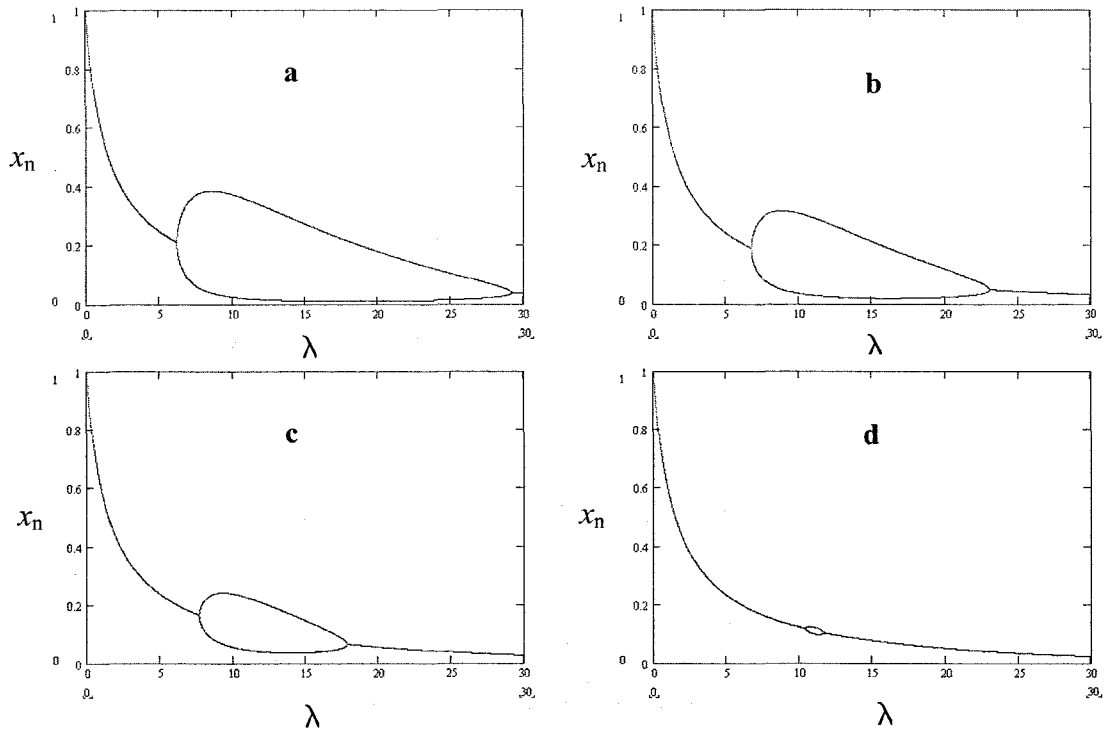


Figure 2

Fig.2a: The curve obtained after iterating eq.(16) for: $a=1.08$ and $b=1$

Fig.2b: The curve obtained after iterating eq.(16) for: $a=1.09$ and $b=1$

Fig.2c: The curve obtained after iterating eq.(16) for: $a=1.10$ and $b=1$

Fig.2d: The curve obtained after iterating eq.(16) for: $a=1.11$ and $b=1$

To confirm these transitions we have calculated numerically the corresponding Lyapunov exponents Λ_j of the maps. For this purpose we have used Mathcad. Doing so, we took into account that the Lyapunov exponent can be estimated, using the formula

$$A = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \ln \left(\left| \frac{dx_{n+1}}{dx_n} \right| \right) \quad (17)$$

which in the case of our map becomes

$$A \approx \frac{1}{N} \sum_{n=1}^N \ln \left(\left| -b \cdot \lambda \cdot (a + x_n)^{-(1+\lambda)} \right| \right) \quad (18)$$

For the calculations of the Lyapunov exponent we have used instead of the general formula (17) the more specific (18). To avoid initial fluctuations we performed the averaging over the last 100 values of 10 000 iterations.

In Figure 3, we demonstrate the bifurcation diagram obtained for $a = 1.09$, $b = 1$ and the corresponding behaviour of the Lyapunov exponents Λ_j . Apparently, at the critical points at which the initial- and reverse- bifurcation occurs, a nullification of the Λ_j 's takes place. In all other regions, the Lyapunov exponents remain negative, indicating a periodic oscillation of the solutions of (9).

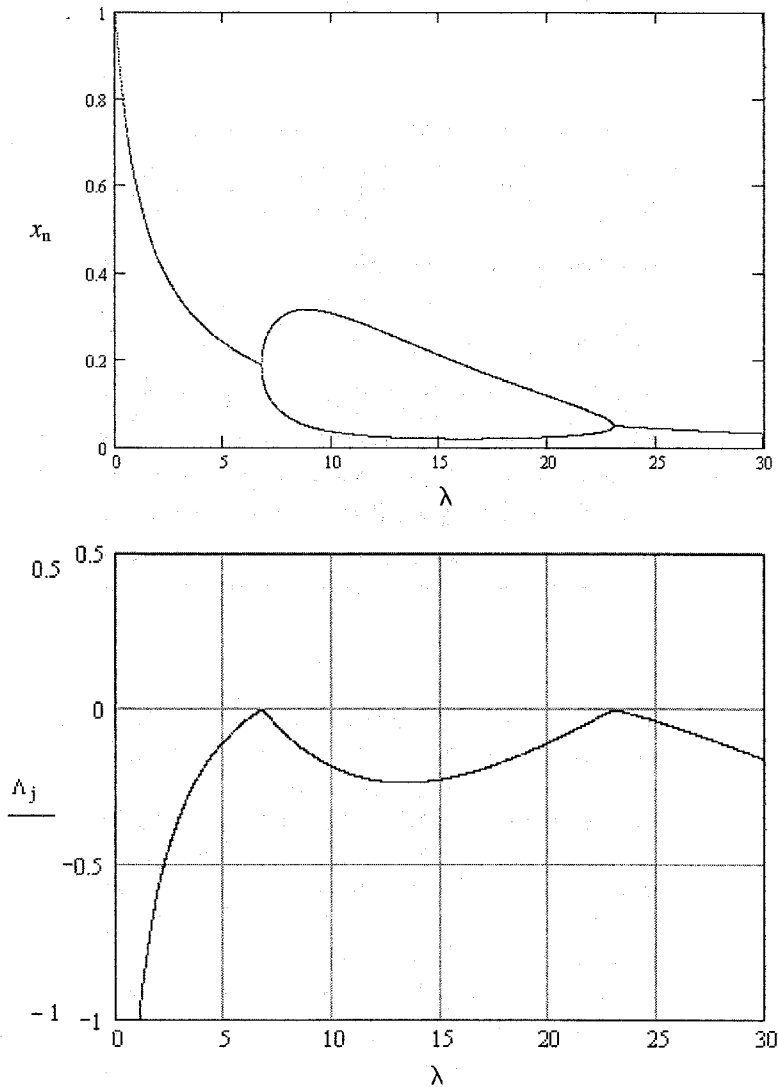


Figure 3

The curve obtained after iterating eq.(16) for: $a=1.09$ and $b=1$ (upper curve) and the corresponding plot of the Lyapunov exponent (lower curve)

4. Comments

The dynamics of the map $f(x) = b(x+a)^{-\lambda}$, are discussed and its bifurcating behaviour is realized for specific values of the control parameter λ . This family of maps is viewed as solution of a differential equation as well as solution of the Relaxed Newton's derivative equal to 2.

The differential equation (9) can be implemented by constructing a non-linear electronic circuit, namely an RLC circuit driven by a sinusoidal voltage. The non-linear component of such a circuit is a capacitor (varactor), whose capacitance varies as a function of the voltage across it, whose form is conjugate to $f(x)$. At very low driving voltages the circuit behaves linearly, exhibiting resonance at a given frequency. By gradually increasing the driving voltage, a non-linear behaviour is observed, resulting in period doubling patterns.

Further investigation in connection with Briot-Bouquet differential equations, is under way.

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Acknowledgments

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