The higher order Schwarzian derivative: Its applications for chaotic behavior and new invariant sufficient condition of chaos

G. Hacibekiroğlu\textsuperscript{a,∗}, M. Çağlar\textsuperscript{b}, Y. Polatoglu\textsuperscript{b}

\textsuperscript{a}Department of Physics, TC İstanbul Kültür University, 34156 İstanbul, Turkey
\textsuperscript{b}Department of Mathematics and Computer Science, TC İstanbul Kültür University, 34156 İstanbul, Turkey

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Abstract

The Schwarzian derivative of a function $f(x)$ which is defined in the interval $(a, b)$ having higher order derivatives is given by

$$S_{f(x)} = \left( \frac{f''(x)}{f'(x)} \right)' - \frac{1}{2} \left( \frac{f''(x)}{f'(x)} \right)^2.$$ 

A sufficient condition for a function to behave chaotically is that its Schwarzian derivative is negative. In this paper, we try to find a sufficient condition for a non-linear dynamical system to behave chaotically. The solution function of this system is a higher degree polynomial. We define the $n$th Schwarzian derivative to examine its general properties. Our analysis shows that the sufficient condition for chaotic behavior of higher order polynomial is provided if its highest order three terms satisfy an inequality which is invariant under the degree of the polynomial and the condition is represented by the Hankel determinant of order 2. Also the $n$th order polynomial can be considered to be the partial sum of real variable analytic function. Let this analytic function be the solution of a non-linear differential equation, then the sufficient condition for the chaotic behavior of this function is that the Hankel determinant of order 2 should be negative, where the elements of this determinant are the coefficients of the terms of $n, n-1, n-2$ in the Taylor expansion.

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1. Introduction

We need the following definition,

\textbf{Definition 1.1.} If $f(x)$ is defined in the interval $(a, b)$ and has higher order derivatives on this interval, then the Schwarzian derivative of $f$ is defined by

$$S_{f(x)} = \left( \frac{f''(x)}{f'(x)} \right)' - \frac{1}{2} \left( \frac{f''(x)}{f'(x)} \right)^2.$$ 

∗ Corresponding author. Tel.: +90 212 498 4316.
E-mail addresses: g.hacibekiroglu@iku.edu.tr (G. Hacibekiroğlu), m.caglar@iku.edu.tr (M. Çağlar), y.polatoglu@iku.edu.tr (Y. Polatoglu).
The equation

\[ \frac{f''''(x)}{f''(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2 = -2\sqrt{f'(x)} \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{f'(x)}} \right) \]

Note that the most important properties of Schwarzian derivative are the following [1,2,6]:

1. The Schwarzian derivative is invariant under the bilinear transformation of \( f(x) \), i.e., if

\[ T(f(x)) = \frac{af(x) + b}{cf(x) + d} \]  

then

\[ S_{T(f(x))} = S_f(x) \]

and there are no transformations possessing this property. We also have

\[ S_{g(t)} = \left( \frac{dx}{dt} \right)^2 S_f(x) + S_{\phi(t)}, \]

where \( x = \phi(t) \), \( f(x) = f(\phi(t)) = g(t) \).

2. The equation \( S_f(x) = 0 \) has the general solution \( f(x) = \frac{ax + b}{cx + d} \) and the following hold:

\[ S_f(x) = -2\sqrt{f'(x)} \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{f'(x)}} \right) = 0 \Rightarrow \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{f'(x)}} \right) = 0 \]

\[ \Rightarrow f'(x) = \frac{1}{(ax + b)^2} \Rightarrow f(x) = \frac{c}{(ax + b)^2}. \]

Now we define the differential operator [3–5,7]

\[ S_{f_n(x)} = \left( \frac{f^{(n+1)}(x)}{f^{(n)}(x)} \right)' - \frac{1}{(n + 1)} \left( \frac{f^{(n+1)}(x)}{f^{(n)}(x)} \right)^2 \]

\[ = \left( \frac{f^{(n+2)}(x)}{f^{(n)}(x)} \right) - \frac{(n + 2)}{(n + 1)} \left( \frac{f^{(n+1)}(x)}{f^{(n)}(x)} \right)^2 \]

\[ = -(n + 1)^{n+1} \sqrt{f^{(n)}(x)} \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{f^{(n)}(x)}} \right) \]

which is called the \( n \)th Schwarzian derivative.

The transformation \( T_{f(x)} \) of \( f(x) \) which leaves its \( n \)th Schwarzian derivative invariant can be obtained in the following way: If \( T_{f(x)} = f_1(x) \) then the relation

\[-(n + 1)^{n+1} \sqrt{f^{(n)}(x)} \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{f^{(n)}(x)}} \right) = -(n + 1)^{n+1} \sqrt{f_1^{(n)}(x)} \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{f_1^{(n)}(x)}} \right) \]

\[ \Rightarrow \frac{f_1^{(n)}(x)}{\sqrt{f_1^{(n)}(x)}} \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{f_1^{(n)}(x)}} \right) = \frac{f^{(n)}(x)}{\sqrt{f^{(n)}(x)}} \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{f^{(n)}(x)}} \right) \]

having defined

\[ u(x) = \frac{1}{\sqrt{f^{(n)}(x)}}, \quad v(x) = \frac{1}{\sqrt{f_1^{(n)}(x)}} \]  

(1.3)
The Schwarzian derivative of the polynomial of
is
(2.1)
and
(1.1)
is given by
(1.6)
we have
Theorem 2.1.
be a polynomial of degree
with 4th order Hankel determinant
Hankel matrix of order 4 of the Fibonacci sequence 1, 1, 2, 3, 5, . . . , is
of order
H.
For
n
and hence
So, the general solution of Eq. (1.4) is given by
(1.5)
where
a
1
and
a
2
are constants. On the other hand from (1.3) and (1.5) we have
(1.6)
and hence

For
n
= 1, (1.6) becomes the bilinear transformation of Eq. (1.1).

The Hankel matrix
H
of the integer sequence
A = \{a_1, a_2, a_3, \ldots\}
is the infinite matrix [8,9]

\[
H = \begin{pmatrix}
  a_1 & a_2 & a_3 & a_4 & \ldots \\
  a_2 & a_3 & a_4 & a_5 & \ldots \\
  a_3 & a_4 & a_5 & a_6 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Therefore the Hankel matrix
H_n
of order of
A
is the upper left \((n \times n)\) sub-matrix of
H,
and
h_n,
the Hankel determinant of order
n
of
A
is the determinant of the corresponding Hankel matrix of order
n,
\(h_n = \det(H_n)\). For example the Hankel matrix of order 4 of the Fibonacci sequence 1, 1, 2, 3, 5, . . . , is

\[
H_4 = \begin{pmatrix}
  1 & 1 & 2 & 3 \\
  1 & 2 & 3 & 5 \\
  2 & 3 & 5 & 8 \\
  3 & 5 & 8 & 13
\end{pmatrix},
\]

with 4th order Hankel determinant \(h_4 = 0\).

2. Main results

In this section we will give a sufficient condition for the chaotic behavior of an \(n\)th degree polynomial. Let \(P(x)\) be a polynomial of degree \(n\) with real coefficient where \(a_1 \neq 0\) having roots \(x_i, i = 1, 2, 3, \ldots, n\)

\[
P(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \cdots + a_{n-1}x + a_n.
\]  

(2.1)

**Theorem 2.1.** The Schwarzian derivative of the polynomial of (2.1) is

\[
S_{P(x)} = \frac{(n + 2)! (n + 1) a_1 \left[ \frac{(n+2)!}{2!} a_1 x^2 + (n + 1)! a_2 x + n! a_3 \right]}{(n + 1) \left[ \frac{(n+2)!}{2!} a_1 x^2 + (n + 1)! a_2 x + n! a_3 \right]^2}
- \frac{(n + 2)(n + 1)! a_1 x + (n + 1)! a_2}{(n + 1) \left[ \frac{(n+2)!}{2!} a_1 x^2 + (n + 1)! a_2 x + n! a_3 \right]^2}.
\]  

(2.2)
Proof. For the proof of (2.2), we will use the induction principle. Using formula (1.2), we get for \( n = 1 \), \( P(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4 \), i.e.,

\[
S_{P(x)} = \left( \frac{P(n+1)(x)}{P(n)(x)} \right)^r - \frac{1}{2} \left( \frac{P(n+1)(x)}{P(n)(x)} \right)^2
\]

\[
\Rightarrow S_{P(x)} = \frac{3!a_1 \left( \frac{3!}{2} a_1 x^2 + 2a_2 x + 1!a_3 \right) - 3(3!a_1 x + 2!a_2 x)^2}{2 \left( \frac{3!}{2} a_1 x^2 + 2!a_2 x + 1!a_3 \right)^2},
\]

so, (2.2) is true for \( n = 1 \).

Suppose that this result is true for \( n = k \). Then we have, \( P(x) = a_1 x^{k+2} + a_2 x^{k+1} + a_3 x^k + a_4 x^{k-1} + \cdots + a_{k+1} \),

\[
S_{P(x)} = \left( \frac{P(k+1)(x)}{P(k)(x)} \right)^r - \frac{1}{k + 1} \left( \frac{P(k+1)(x)}{P(k)(x)} \right)^2
\]

\[
\Rightarrow S_{P(x)} = \frac{(k + 2)! (k + 1) a_1 \left( \frac{(k + 2)!}{2!} a_1 x^2 + (k + 1)! a_2 x + k! a_3 \right)}{(k + 1) \left( \frac{(k + 2)!}{2!} a_1 x^2 + (k + 1)! a_2 x + k! a_3 \right)^2}
\]

\[
- \frac{(k + 2) ((k + 2)! a_1 x + (k + 1)! a_2)^2}{(k + 1) \left( \frac{(k + 2)!}{2!} a_1 x^2 + (k + 1)! a_2 x + k! a_3 \right)^2},
\]

From the induction hypothesis, we then have, \( P(x) = a_1 x^{k+3} + a_2 x^{k+2} + a_3 x^{k+1} + \cdots + a_{k+1} \),

\[
S_{P(x)} = \left( \frac{P(k+2)(x)}{P(k+1)(x)} \right)^r - \frac{1}{k + 2} \left( \frac{P(k+2)(x)}{P(k+1)(x)} \right)^2
\]

\[
\Rightarrow S_{P(x)} = \frac{(k + 3)! (k + 2) a_1 \left( \frac{(k + 3)!}{2!} a_1 x^2 + (k + 2)! a_2 x + (k + 1)! a_3 \right)}{(k + 2) \left( \frac{(k + 3)!}{2!} a_1 x^2 + (k + 2)! a_2 x + (k + 1)! a_3 \right)^2}
\]

\[
- \frac{(k + 3) ((k + 3)! a_1 x + (k + 2)! a_2)^2}{(k + 2) \left( \frac{(k + 3)!}{2!} a_1 x^2 + (k + 2)! a_2 x + (k + 1)! a_3 \right)^2},
\]

which is the desired conclusion. \( \Box \)

Theorem 2.2. A necessary condition for an nth degree polynomial to behave chaotically is that

\[
a_2^2 - \frac{2n + 4}{2n + 3} a_1 a_3 > 0 \quad \text{for all } n = 1, 2, 3, \ldots.
\]

Proof. Using the result of Theorem 2.1, we get

\[
S_{P(x)} < 0 \Rightarrow \left[ \frac{(n + 2)! (n + 1)}{2!} - (n + 2)! (n + 2) \right] a_1^2 x^2
\]

\[+[(n + 1)! (n + 2)! (n + 1) - 2(n + 1)! (n + 3)! (n + 2)!] a_1 a_2 x \]

\[+[(n)! (n + 2)! (n + 1) a_1 a_3 - ((n + 1)!)^2 (n + 2) a_2^2] < 0 \]

or

\[
\left[ \frac{(n + 2)! (n + 2)}{2!} - \frac{(n + 2)! (n + 1)}{2!} \right] a_1^2 x^2
\]

\[\quad + [2(n + 1)! (n + 2)! (n + 2) - (n + 1)! (n + 2)! (n + 1)] a_1 a_2 x^2 \]

\[\quad + [(n + 1)! (n + 2)! (n + 2) a_2^2 - (n)! (n + 2)! (n + 1) a_1 a_3] > 0. \quad (2.3)
\]
If we take
\[ A = \left( (n+2)! \right)^2 (n+2) - \frac{(n+2)!^2 (n+1)}{2!} \right] a_1^2, \]
\[ B = [2(n+1)! (n+2)! (n+2) - (n+1)! (n+2)! (n+1)] a_1 a_2, \]
\[ C = [(n+1)!] n!(n+2) a_2^2 - (n)!(n+2)! (n+1) a_1 a_3, \]
then the inequality (2.3) can be written in the form
\[ Ax^2 + Bx + C > 0. \] 
(2.4)
In order for condition (2.4) to be satisfied, the discriminant of the polynomial should be negative and the coefficient of \( x^2 \) should be positive, i.e.,
\[ A = \left( (n+2)! \right)^2 (n+2) - \frac{(n+2)!^2 (n+1)}{2!} \right] a_1^2 > 0 \] 
(2.5)
i.e.,
\[ B^2 - 4AC = [(n+2)! (n+3) - 2(n+2)! (n+3) a_2 - 2(n+2)! (n+3) a_1 a_3] < 0. \]
If we write (2.5) for \( n = 1, 2, 3, \ldots \) we obtain the following inequalities
\[ n = 1, \quad a_2^2 - 3 a_1 a_3 > 0 \Rightarrow a_2^2 - 3 a_1 a_3 > 0, \]
\[ n = 2, \quad 7a_2^2 - 8a_1 a_3 > 0 \Rightarrow a_2^2 - \frac{8}{7} a_1 a_3 > 0, \]
\[ n = 3, \quad 9a_2^2 - 10a_1 a_3 > 0 \Rightarrow a_2^2 - \frac{10}{9} a_1 a_3 > 0, \]
\[ n = 4, \quad 11a_2^2 - 12a_1 a_3 > 0 \Rightarrow a_2^2 - \frac{12}{11} a_1 a_3 > 0, \]
\[ n = 5, \quad 13a_2^2 - 14a_1 a_3 > 0 \Rightarrow a_2^2 - \frac{14}{13} a_1 a_3 > 0, \]
\[ \vdots \]
so, by induction, we have
\[ a_2^2 - \frac{2n+4}{2n+3} a_1 a_3 > 0, \quad n \geq 2. \]
\[ \square \]
As a result, a necessary condition for the chaotic behavior of the polynomial \( P(x) \), is that
\[ a_2^2 - \frac{2n+4}{2n+3} a_1 a_3 > 0, \quad n \geq 2. \]
Furthermore, letting \( n \) go to infinity, we have \( a_2^2 - a_1 a_3 > 0 \). This implies that a real analytic function whose Maclaurin series expansion is \( \sum_{n=0}^{\infty} a_n x^n \) behaves chaotically whenever \( a_2^2 - a_1 a_3 > 0 \). If we use the Hankel determinant, this condition can be written in the form
\[ \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} < 0. \]

References